On a property of discrete sets in $\mathbb{R}^k$

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We consider discrete sets in $\mathbb{R}^k$ where each point has a finite multiplicity. We call them *discrete multiple sets*. For any two discrete multiple sets $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ we define a *distance* between them in such a way:

$$dist(\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}) = \inf_{\sigma} \sup_{n\in\mathbb{N}} |a_n - b_{\sigma(n)}|,$$

where infimum is taken over all bijections $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$. This function satisfies all the axioms of metric except the finiteness.

Definition 1. *A vector $\tau \in \mathbb{R}^k$ is called an ε-almost period of a discrete multiple set $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^k$, if*

$$dist(\{a_n\}_{n\in\mathbb{N}}, \{a_n + \tau\}_{n\in\mathbb{N}}) < \varepsilon.$$

Definition 2. *A discrete multiple set $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^k$ is called almost periodic, if for each $\varepsilon > 0$ the set of its ε-almost periods is relatively dense in $\mathbb{R}^k$.*

A simple example of an almost periodic set is the set

$$\{\alpha m + F(m)\}_{m\in\mathbb{Z}^k}$$

where $F(m)$ is an almost periodic mapping from $\mathbb{Z}^k$ to $\mathbb{R}^k$, $\alpha > 0$.

For each almost periodic multiple set D there exists $M < \infty$ such that for all $c \in \mathbb{R}^k$ card $(D \cap \{x : \|x - c\| < 1) < M$. The limit of almost periodic multiple sets is almost periodic as well. For such sets some analogue of the Bochner criterion of almost periodicity holds. Any almost periodic multiple set D possesses finite nonzero shift invariant density

$$\Delta = \lim_{T\to\infty} \frac{\text{card } (D \cap \{x : |x_1| < T, \ldots, |x_k| < T\})}{(2T)^k}.$$

Next we prove that our definition of an almost periodic set is equivalent to the classical one: the discrete set is almost periodic if the measure with unit masses in points of the set is almost periodic in a weak sense.

An almost periodic set is one of models describing quasicristallic structures (see [1]).

References

[1] J.C. Lagarias, Mathematical quasicristals and the problem of diffraction, Directions in Mathematical Quasicrustals, M. Baake and R. Moody, eds., CRM Monograph series, Vol. 13, AMS, Providence RI, 2000, 61-93.

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