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Кваліфікаційна робота
Освітньо-кваліфікаційний рівень: магістр
за темою: "Розкладення дії симетричної групи на повній матричній алгебрі по незвідним компонентам."

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Харків - 2025

1 Introduction

For the finite symmetrical groups S_n there exists the full description of the irreducible representations. The first results in this direction were obtained by Schur (1911) who found their parametrization by the partitions or Young diagrams. There are many interesting natural representations of the groups S_n that are not irreducible. The question of how a given representation can be decomposed into a direct sum of the irreducible components is often discussed. The corresponding problem is called harmonic analysis on the group.

The group S_n is realized by the permutations (bijections) of the set $\{1, 2, \ldots, n\}$. In the case $n = \infty$ there exists two analogous of the finite symmetrical groups:

- the group S_∞ of the finite[1] bijections of the all natural numbers $\mathbb{N}$;
- the group $\widetilde{S}_\infty$ of all bijection of $\mathbb{N}$.

1.1 The results

The aim of this thesis to obtain the explicitly decomposition of the action of S_n on the full matrix algebra $\mathrm{Mat}(n, \mathbb{C})$ by the conjugations. Namely, group S_n is embedded in the group $U(n, \mathbb{C})$ of the unitary matrices:

$$S_n \ni s \overset{\mathfrak{i}_n}{\mapsto} \left[\delta_{i\, s(j)}\right] \in U(n, \mathbb{C}), \quad \text{where } \delta \text{ is Kronecker delta.}$$

It is easy to check that the mapping $\mathfrak{i}_n$ is a homomorphism. Therefore, the mapping

$$S_n \times \mathrm{Mat}(n, \mathbb{C}) \ni (s, m) \mapsto \left[\delta_{i\, s(j)}\right] \; m \; \left[\delta_{i\, s^{-1}(j)}\right] = T(s)m \in \mathrm{Mat}(n, \mathbb{C})$$

is the representation of S_n.

We found all irreducible components of T.

There exist the following T- invariant subspaces[2]:

- the subspace $\mathrm{Mat}_{sym}(n, \mathbb{C})$ of the symmetric matrices,

$$\mathrm{Mat}_{sym}(n, \mathbb{C}) = \{M = [M_{ij}] : M_{ij} = M_{ji}]\};$$

  $\mathrm{Mat}_{sym}(n, \mathbb{C})$ is the direct sum of the next irreducible components:

[1] bijection s is finite if the set $\{k \in \mathbb{N} : s(k) \neq k\}$ is finite.

[2] a subspace $E \subset \mathrm{Mat}(n, \mathbb{C})$ is T- invariant if $T(S_n)E \subset E$

- 1) n-dimensional subspace of the diagonal matrices $\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$ is the direct sum of the following irreducible components:
  - a) one dimensional subspace of the scalar matrices,
  - b) $(n-1)$-dimensional subspace of the diagonal matrices such that $\sum\limits_{k=1}^{n} \lambda_k = 0$;
- 2) n-dimensional subspace of the matrices of the form $B = [B_{ij}]$, where $B_{ij} = b_i + b_j$ for all $i \neq j$ and $B_{ii} = 0$, is the linear span of the matrices

$$E_1 = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix}, ...E_n = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}$$

  and it is decomposed into the direct sum of two irreducible components:
  - a) one dimensional subspace of the view $c \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}$
  - b) $(n-1)$-dimensional irreducible subspace of the matrices $B = [B_{ij}]$, where $B_{ij} = b_i + b_j$ for all $i \neq j$ and $B_{ii} = 0$, such that $\sum\limits_{k=1}^{n} b_k = 0$;
- 3) the irreducible $\left(\frac{n^2-3n}{2}\right)$-dimensional subspace $\widetilde{\mathrm{Mat}}_{sym}(n, \mathbb{C}) \subset \mathrm{Mat}_{sym}(n, \mathbb{C})$ of the matrices $M = [M_{ij}]$ such that $M_{ii} = 0$ and $\sum\limits_{i=1}^{m} M_{ij} = 0$ for all j:

• the subspace $\mathrm{Mat}_{asym}(n, \mathbb{C}) = \{M = [M_{ij}] \in \mathrm{Mat}(n, \mathbb{C}) : M_{ij} = -M_{ji}\}$ of the antisymmetric (skew-symmetric) matrices has the next decomposition into the direct sum of the irreducible components:

- 1) $(n-1)$-dimensional irreducible subspace of the matrices of the form $B = [B_{ij}]$, where $B_{ij} = b_i - b_j$ for all $i \neq j$ and $B_{ii} = 0$, is the linear span of the matrices

$$E_1 = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ -1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & -1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 1 \\ 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \end{bmatrix}, \cdots$$

$$E_n = \begin{bmatrix} 0 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & -1 \\ \vdots & \vdots & \vdots & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}$$

– 2) the irreducible $\left(\frac{n^2-3n+2}{2}\right)$-dimensional subspace $\widetilde{\mathrm{Mat}}_{asym}(n, \mathbb{C}) \subset \mathrm{Mat}_{asym}(n, \mathbb{C})$ of the matrices $M = [M_{ij}]$ such that $\sum\limits_{i=1}^{m} M_{ij} = 0$ for all j.

The above result extended on the case $n = \infty$. Namely, we consider the infinity dimensional space $\mathrm{Mat}(HS)$ of the complex Hilbert-Schmidt matrices instead $\mathrm{Mat}(n, \mathbb{C})$. We recall that

$$\mathrm{Mat}(HS) = \left\{ M = [M_{ij}] : \sum_{i,j=1}^{\infty} |M_{ij}|^2 < \infty \right\}.$$

Then the action of S_∞ on $\mathrm{Mat}(HS)$ by the conjugations is decomposed into the direct sum of the following irreducible components:

- the subspace of the diagonal matrices;
- the subspace of the symmetric matrices $M = [m_{ij}]$ such that $M_{ii} = 0$ for all i;
- the subspace of the skew-symmetric matrices.

1.2 Main definitions and notions.

Definition 1. A representation of a finite group G in a finite-dimensional complex vector space V is a homomorphism ρ: $G \to GL(V)$ of G to the group of automorphisms of V.

Definition 2. Subspace W is invariant under ρ if $\rho(g)W \subset W$ for all $g \in G$.

Definition 3. A representation ρ is called **irreducible** if there is no proper nonzero invariant subspace W of V.

Definition 4. $Lin\{v_1, ..., v_n\}$ is the smallest linear subspace that contains vectors $v_1, ..., v_n \in V$.

Lemma 5. *$Lin\{\rho(G)v\}$ is invariant under ρ, for every nonzero $v \in V$, $dim(V) < \infty$.*

Proof. Let $a \in Lin\{\rho(G)v\}, \overline{g} \in G, v \in V$, then $\rho(\overline{g})(a) \in Lin\{\rho(G)v\}$. Since $a \in Lin\{\rho(G)v\}$, then $a = \lambda_1\rho(g_1)(v) + ... + \lambda_n\rho(g_n)(v)$, where $g_1, ..., g_n \in G$, $\lambda_1, ..., \lambda_n \in \mathbb{C}$. Applying ρ to a, we obtain: $\rho(\overline{g})(a) = \rho(\overline{g})(\lambda_1\rho(g_1)(v) + ... + \lambda_n\rho(g_n)(v))$.

Since $\rho(\overline{g})(a)$ is a linear operator on V, then the next equality holds : $\rho(\overline{g})(a) = \lambda_1\rho(\overline{g})(\rho(g_1)(v)) + ... + \lambda_n\rho(\overline{g})(\rho(g_n)(v))$. Using properties of homomorphism ρ, we obtain that $\rho(\overline{g})(a) = \lambda_1\rho(\overline{g}g_1)(v) + ... + \lambda_n\rho(\overline{g}g_n)(v)$. Since $\overline{g}g_1, ..., \overline{g}g_n \in G$, then $\rho(\overline{g})(a) \in Lin\{\rho(G)v\}$. □

Lemma 6. *A representation ρ is irreducible if and only if for any nonzero $v \in V$ the equality $Lin\{\rho(G)v\} = V$ holds.*

Proof. Using Lemma 5, we obtain that $Lin\{\rho(G)v\}$ is an invariant subspace for any v. Assume that ρ is irreducible. Since $Lin\{\rho(G)v\}$ is invariant, by definition 3, we have equality $Lin\{\rho(G)v\} = V$ for all $v \neq 0$.

Let invariant subspace $W \subset V$, $W \neq V$. Take nonzero $w \in W$. By an invariance, $Lin\{\rho(G)w\} \subset W$. But it follows from cyclicity that the next equality holds: $Lin\{\rho(G)w\} = V$. Thus, $V \subset W$. We have a contradiction. □

2 Action of S_n on a complex vector space by the permutations of coordinates

Here we consider examples of representations of the permutation group of a finite set $\{1, 2, ..., n\}$. We denote this group by S_n.
Let $\mathbb{C}^n$ be an ordinary complex vector space. Suppose that the elements of $\mathbb{C}^n$ form a row $v = (v_1, v_2, ..., v_n)$, where $v_1, v_2, ..., v_n \in \mathbb{C}$.
Define the representation π of S_n as follows:

$$\pi(g)(v) = (v_{g(1)}, v_{g(2)}, ..., v_{g(n)}),\ g \in S_n. \tag{2.1}$$

Let‘s decompose V into a direct sum of irreducible subrepresentations.

Obviously that $V_1 = \{v \in \mathbb{C}^n : v = (c, ..., c), c \in \mathbb{C}\}$ is an invariant subspace. Namely, we have $\pi(g)(v) = v$ for all $v \in V_1$ and $g \in S_n$. It is clear that $dim(V_1) = 1$. Thus, V_1 is a trivial case of an irreducible subrepresentation.

Let‘s find the orthogonal complement to V_1. $V_1^{\perp} = \{v \in V : \langle v, w\rangle = 0, w \in V_1\}$, where $\langle\cdot, \cdot\rangle$ is a standart inner product on $\mathbb{C}^n$. Hence

$$\langle v, w\rangle = \sum_{i=1}^{n} v_i \cdot \overline{w_i} = 0, \text{where } \overline{w_i} \text{ is a complex conjugate of } w_i.$$

If $w = (c, ..., c)$, then $\overline{c}\sum_{i=1}^{n} v_i = 0$. Thus, $\pi(g)(V_1^{\perp}) = V_1^{\perp}\ \forall g \in S_n$.

Proposition 7. *The restriction of π to $V_1^{\perp}$ is an irreducible representation.*

Proof. Let‘s take an arbitrary nonzero $a \in V_1^{\perp}$, $a = (a_1, a_2, ..., a_n)$. If $a_1 \neq 0$ or $a_2 \neq 0$ and $a_1 \neq a_2$, then we have:

$f(a) = \frac{1}{a_1 - a_2}(a - \pi((1\ 2))(a)) = (1, -1, 0, ..., 0)$.

Let’s consider the case when $a_1 = a_2 = 0$. Since $a \neq 0$, there exists such $2 < l \leq n$ that $a_l \neq 0$. Then apply $f(\pi((1\ l))(a)) = (1, -1, 0, ..., 0)$.
Let‘s consider the case when $a_1 = a_2 \neq 0$. Since $a \in V_1^{\perp}$, there exists such $2 < l \leq n$ that $a_l \neq a_1$. Then $f(\pi((1\ l))(a)) = (1, -1, 0, ..., 0)$.
Let‘s consider vectors $e_1, ..., e_2$ of the form:

$e_1 = \pi((2\ n))((1, -1, 0, ..., 0)) = (1, 0, 0, ..., -1)$,

$e_2 = \pi((1\ 2))(e_1) = (0, 1, 0, ..., -1)$,

...

$e_i = \pi((1\ i))(e_1) = (0, ..., 0, \underbrace{1}_{i}, 0, ..., -1)$,

...

$e_{n-1} = \pi((1\ n-1))(e_1) = (0, ..., 0, 1, -1)$.

Let‘s prove that $e_1, ..., e_{n-1}$ are linearly independent.
Suppose there exist $\lambda_1, ..., \lambda_{n-1} \in \mathbb{C}$ such that $\lambda_1 e_1 + ... + \lambda_{n-1} e_{n-1} = 0$. Thus, $\lambda_1 e_1 + ... + \lambda_{n-1} e_{n-1} = (\lambda_1, ..., \lambda_{n-1}, -\sum_{i=1}^{n-1} \lambda_i) = 0$. Thus $\lambda_1 = ... = \lambda_{n-1} = 0$. Since $e_1, ..., e_{n-1}$ are linearly independent and $|\{e_1, ..., e_{n-1}\}| = n-1$, then the system of vectors $e_1, ..., e_{n-1}$ is a basis in the space $V_1^{\perp}$. Thus, the restriction of π to $V_1^{\perp}$ is an irreducible representation.

□

3 Action of S_n on a full matrix algebra $Mat(n, \mathbb{C})$ by conjugation

Denote by $U(n, \mathbb{C})$ the unitary subgroup of $Mat(n, \mathbb{C})$. Define the embedding of S_n into $U(n, \mathbb{C})$ by

$$S_n \ni s \to [\delta_{i\, s(j)}] \in U(n, \mathbb{C}), \tag{3.2}$$

where $\delta_{i\,j} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$ is Kronecker delta.

Since

$$[\delta_{i\ s(j)}][\delta_{k\ t(l)}] = [\sum_{r=1}^{n} \delta_{i\ s(r)} \delta_{r\ t(l)}] = [\delta_{i\ s(t(l))}],$$

we have

$$st \to [\delta_{i\ s(t(l))}] = [\delta_{i\ s(j)}][\delta_{k\ t(l)}].$$

Thus embedding (3.2) is homomorphism.

For convenience we identify the elements $s \in S_n$ with the corresponding matrix $[\delta_{i\ s(j)}]$.

Let's define representation ρ by

$$\rho(s)(v) = [\delta_{i\ s(j)}] \cdot v \cdot [\delta_{i\ s(j)}]^{-1}, \text{ where } v \in Mat(n, \mathbb{C}), s \in S_n.$$

Let's define an inner product on $Mat(n, \mathbb{C})$ by

$$\langle A, B \rangle = \mathrm{Tr}(A \cdot B^*), \text{ where } A, B \in Mat(n, \mathbb{C})$$

Direct verification shows that

$$\mathrm{Tr}(A \cdot B^*) = \mathrm{Tr}(B^* \cdot A).$$

Lemma 8. $\langle A, B \rangle = \langle \rho(s)(A), \rho(s)(B) \rangle$.

Proof. $\mathrm{Tr}(\rho(s)(A) \cdot (\rho(s)(B))^*)$
$= \mathrm{Tr}([\delta_{i\ s(j)}] \cdot A \cdot [\delta_{i\ s(j)}]^{-1} \cdot ([\delta_{i\ s(j)}] \cdot B \cdot [\delta_{i\ s(j)}]^{-1})^*)$
$= \mathrm{Tr}([\delta_{i\ s(j)}] \cdot A \cdot [\delta_{i\ s(j)}]^{-1} \cdot ([\delta_{i\ s(j)}]^{-1})^* \cdot B^* \cdot ([\delta_{i\ s(j)}])^*)$
$= \mathrm{Tr}([\delta_{i\ s(j)}] \cdot A \cdot B^* \cdot ([\delta_{i\ s(j)}])^*)$
$= \mathrm{Tr}(B^* \cdot [\delta_{i\ s(j)}]^* \cdot [\delta_{i\ s(j)}] \cdot A) = \mathrm{Tr}(B^* \cdot A)$
$= \mathrm{Tr}(A \cdot B^*)$. □

Thus, ρ is an unitary representation of S_n.

3.1 ρ-invariant subspaces of $Mat(n, \mathbb{C})$

Define subspaces:

$$Mat_s(n, \mathbb{C}) = \{M \in Mat(n, \mathbb{C}) : M = M^t\},$$

$$Mat_{as}(n, \mathbb{C}) = \{M \in Mat(n, \mathbb{C}) : M = -M^t\},$$

where M^t is the matrix transposed to matrix M.

Lemma 9. *Any $m \in Mat(n, \mathbb{C})$ can be represented as $m = m_s + m_{as}$, where $m_s \in Mat_s(n, \mathbb{C})$, $m_{as} \in Mat_{as}(n, \mathbb{C})$.*

Proof. Since $m \in Mat(n, \mathbb{C})$, then $m^t \in Mat(n, \mathbb{C})$. Thus, every $m \in Mat(n, \mathbb{C})$ can be represented as $m = m_s + m_{as}$, where $m_s = \frac{m+m^t}{2}$, $m_{as} = \frac{m-m^t}{2}$. □

Lemma 10. *$Mat_s(n, \mathbb{C})$ and $Mat_{as}(n, \mathbb{C})$ are ρ-invariant.*

Proof. Let $m_1 \in Mat_s(n, \mathbb{C})$. By definition of ρ, we have

$$\rho(s)(m_1) = [\delta_{i\ s(j)}] \cdot m_1 \cdot [\delta_{i\ s(j)}]^{-1}, and$$

$$(\rho(s)(m_1))^t = ([\delta_{i\ s(j)}]^{-1})^t \cdot m_1^t \cdot [\delta_{i\ s(j)}]^t = [\delta_{i\ s(j)}] \cdot m_1 \cdot [\delta_{i\ s(j)}]^{-1} = \rho(s)(m_1).$$

Let $m_2 \in Mat_{as}(n, \mathbb{C})$. By definition of ρ, we have

$$\rho(s)(m_2) = [\delta_{i\ s(j)}] \cdot m_2 \cdot [\delta_{i\ s(j)}]^{-1}, and$$

$$(\rho(s)(m_2))^t = -([\delta_{i\ s(j)}]^{-1})^t \cdot m_2^t \cdot [\delta_{i\ s(j)}]^t = -[\delta_{i\ s(j)}] \cdot m_2 \cdot [\delta_{i\ s(j)}]^{-1} = -\rho(s)(m_2).$$

□

Obviously that $Mat_s(n,\mathbb{C}) \cap Mat_{as}(n,\mathbb{C}) = 0$

Lemma 11. *$Mat_s(n,\mathbb{C})$ and $Mat_{as}(n,\mathbb{C})$ are mutually orthogonal.*

Proof. Let's take $A \in Mat_s(n,\mathbb{C})$ and $B \in Mat_{as}(n,\mathbb{C})$. Then $\langle A, B\rangle = \mathrm{Tr}(A \cdot B^*) = \mathrm{Tr}((A \cdot B^*)^t) = \mathrm{Tr}((B^*)^t \cdot A^t) = -\mathrm{Tr}(B^* \cdot A) = -\mathrm{Tr}(A \cdot B^*)$. Thus $\mathrm{Tr}(A \cdot B^*) = 0$. □

Remark 12. $\dim(Mat_s(n,\mathbb{C})) = \frac{n^2+n}{2}$, $\dim(Mat_{as}(n,\mathbb{C})) = \frac{n^2-n}{2}$

3.2 Irreducible components of $Mat_s(n,\mathbb{C})$

Denote by D the subspace of all diagonal matrices:

$$\mathrm{diag}(\lambda_1, ..., \lambda_n) = \begin{bmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & \lambda_n \end{bmatrix}.$$

It is clear that the vectors from $\mathbb{C}E$ are ρ-invariant, where E is a unit matrix - $\mathrm{diag}(1, ..., 1)$. Namely, $\rho(s)(m) = m$ for all $m \in \mathbb{C}E$.

Let's consider $D_0 = \{\mathrm{diag}(\lambda_1, ..., \lambda_n) : \Sigma_{i=1}^n \lambda_i = 0\}$. It's clear that $D = D_0 \oplus \mathbb{C}E$.

Denote by ρ_D the restriction of ρ to the subspace D.

Proposition 13. *D_0 and $\mathbb{C}E$ are ρ_D-invariant, and restrictions of ρ_D to D_0 and $\mathbb{C}E$ are irreducible.*

Proof. Let's define $f : \mathrm{diag}(\lambda_1, ..., \lambda_n) \to \mathbb{C}^n$, by

$$f\left(\begin{bmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & \lambda_n \end{bmatrix}\right) = (\lambda_1, ..., \lambda_n).$$

Note that f is an intertwining operator between the representations ρ_D and π (see (2.1)). Actually f is a bijection. Since $f(cE) = (c, ..., c)$ and $f(D_0) = V_1^{\perp}$ (see page 4), then, using the Proposition 7, we obtain that the restriction of ρ_D to D_0 is irreducible. Obviously, $\mathbb{C}E$ is irreducible. Thus, the restrictions of ρ_D to D_0 and $\mathbb{C}E$ are irreducible representations. □

Let's introduce the subspace of matrices $Lin\{e_i^s\}_{i=1}^n$, where

$$e_i^s = [(e_i^s)_{k\,l}] = \begin{bmatrix} 0 & ... & 0 & 1 & 0 & ... & 0 \\ 0 & ... & 0 & 1 & 0 & ... & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & ... & 1 & 0 & 1 & ... & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & ... & 0 & 1 & 0 & ... & 0 \end{bmatrix}.$$

Namely, if $l = k$ then $(e_i^s)_{k\,l} = 0$, if $l \neq k$ then $(e_i^s)_{k\,l} = 1$ in the case $k = i$ or $l = i$, in opposite case $(e_i^s)_{k\,l} = 0$.

Remark 14. If $n = 2$ then $e_1^s = e_2^s$. In the case $n > 2$, we obtain that $e_i^s \neq e_j^s$ for all $i \neq j$.

Lemma 15. *For $n > 2$ collection $\{e_i^s\}_{i=1}^n$ is a set of linearly independent vectors.*

Proof. Let's assume there exist the numbers $\lambda_1, \lambda_2, ..., \lambda_n$ such that

$$\sum_{i=1}^{n} \lambda_i e_i^s = \begin{bmatrix} 0 & \lambda_1 + \lambda_2 & ... & \lambda_1 + \lambda_n \\ \lambda_2 + \lambda_1 & 0 & ... & \lambda_2 + \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_n + \lambda_1 & \lambda_n + \lambda_2 & ... & 0 \end{bmatrix} = 0.$$

Thus $-\lambda_1 = \lambda_2 = \lambda_3 = ... = \lambda_n$ and $\lambda_n = -\lambda_2$. Since $-\lambda_2 = \lambda_2$, then $\lambda_2 = 0$ and $\lambda_1 = \lambda_2 = ... = \lambda_n = 0$. □

Let's denote $Lin\{\{e_i^s\}_{i=1}^n\}$ as X_s.

Proposition 16. *Let's introduce the subspaces $X_s^c = \{\lambda \sum_{i=1}^{n} e_i^s\}_{\lambda \in \mathbb{C}}$, and $X_s^0 = \{\sum_{i=1}^{n} \lambda_i e_i^s : \sum_{i=1}^{n} \lambda_i = 0\}_{\lambda \in \mathbb{C}}$. Then X_s^c and X_s^0 are ρ-invariant and the restrictions of ρ to X_s^c and X_s^0 are irreducible representations.*

Proof. Consider an operator

$$\mathcal{F} : X_s \ni e_i^s \to (\underbrace{0, ..., 0}_{i-1}, 1, 0, ..., 0) \in \mathbb{C}^n$$

$\mathcal{F}$ is an intertwining operator and irreducibility is invariant under equivalent representations. Applying Proposition 7, we obtain that restrictions of ρ to X_s^c and X_s^0 are irreducible representations. □

Let's denote by $\widetilde{Mat_s}(n, \mathbb{C})$ the subspace of symmetric matrices $[m_{ij}]$ such that $m_{ii} = 0$ and $\sum_{j=1}^n m_{ij} = 0$ for all $0 < i \leq n$.

Proposition 17. *Subspace $Mat_s(n, \mathbb{C})$ is an orthogonal sum of $\mathbb{C}E \oplus D_0 \oplus X_s \oplus \widetilde{Mat_s}(n, \mathbb{C})$.*

Proof. Direct verification shows that $\mathbb{C}E \oplus D_0 \oplus X_s \oplus \widetilde{Mat_s}(n, \mathbb{C})$ are mutually orthogonal.

Let's prove that $X_s \perp \widetilde{Mat_s}(n, \mathbb{C})$. Let's take $v \in \widetilde{Mat_s}(n, \mathbb{C})$, then $\langle v, e_i^s \rangle = 2 \sum_{j=1}^n v_{ij} = 0$, where $e_i^s \in X_s$ for any $i \in \overline{1, n}$. Let's take an arbitrary $w \in X_s$. w can be represented as $w = w_1 e_1^s + ... + w_n e_n^s$. Thus, for any $v \in \widetilde{Mat_s}(n, \mathbb{C})$. $\langle v, w \rangle = w_1 \langle v, e_1^s \rangle + ... + w_n \langle v, e_n^s \rangle = 0$.

$\mathbb{C}E$ and D_0 are mutually orthogonal and $(\mathbb{C}E \oplus D_0)$ and $X_s \oplus \widetilde{Mat_s}(n, \mathbb{C})$ are mutually orthogonal.

Suppose there exists such nonzero vector $A \in Mat_s(n,\mathbb{C})$ such that A is orthogonal to $\mathbb{C}E \oplus D_0 \oplus X_s \oplus \widetilde{Mat_s}(n,\mathbb{C})$. It follows from orthogonality to $\mathbb{C}E \oplus D_0$ that $a_{ii} = 0$ for each $i \in \overline{1,n}$. Considering orthogonality to X_s we obtain that $\Sigma_{j=1}^{n} a_{ij} = 0$ for each $i \in \overline{1,n}$. Hence, $A \in \widetilde{Mat_s}(n,\mathbb{C})$ and A is orthogonal to each element from $\widetilde{Mat_s}(n,\mathbb{C})$, including A. Since $\langle A, A\rangle = 0$, then $A = 0$. We obtained a contradiction. Thus $Mat_s(n,\mathbb{C}) = \mathbb{C}E \oplus D_0 \oplus X_s \oplus \widetilde{Mat_s}(n,\mathbb{C})$. □

Remark 18.

$$dim(\widetilde{Mat_s}(n,\mathbb{C})) = \frac{n^2-n}{2} - n = \frac{n^2-3n}{2}.$$

Let's introduce matrices of the view:

$$E^{2\,j} = [e^{2\,j}_{k\,l}] = \begin{bmatrix} 0 & 0 & 1 & ... & -1 & ... & 0 \\ 0 & 0 & -1 & ... & 1 & ... & 0 \\ 1 & -1 & 0 & 0 & 0 & ... & 0 \\ \vdots & \vdots & 0 & 0 & 0 & ... & 0 \\ -1 & 1 & 0 & 0 & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & ... & 0 \end{bmatrix}, \text{where}\, j > 3.$$

Namely, $e^{2\,j}_{1\,3} = e^{2\,j}_{3\,1} = e^{2\,j}_{2\,j} = e^{2\,j}_{j\,2} = 1$, $e^{2\,j}_{2\,3} = e^{2\,j}_{3\,2} = e^{2\,j}_{1\,j} = e^{2\,j}_{j\,1} = -1$. Otherwise, $e^{2\,j}_{k\,l} = 0$.

Note that the elements $e^{2\,j}_{k\,l}$ for $k,l \in \overline{1,3}$ are the same for all matrix $E^{2\,j}$, where $j > 3$. Note that $e^{2\,j}_{2\,j} = e^{2\,j}_{j\,2} = 1$.

$$E^{3\,j} = [e^{3\,j}_{k\,l}] = \begin{bmatrix} 0 & 1 & 0 & ... & -1 & ... & 0 \\ 1 & 0 & -1 & ... & 0 & ... & 0 \\ 0 & -1 & 0 & ... & 1 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & ... & 0 \\ -1 & 0 & 1 & \vdots & \ddots & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & ... & 0 \end{bmatrix}, \text{where}\, j > 3.$$

Namely, $e^{3\,j}_{1\,2} = e^{3\,j}_{2\,1} = e^{3\,j}_{3\,j} = e^{3\,j}_{j\,3} = 1$, $e^{3\,j}_{2\,3} = e^{3\,j}_{3\,2} = e^{3\,j}_{1\,j} = e^{3\,j}_{j\,1} = -1$. Otherwise, $e^{3\,j}_{k\,l} = 0$.

Note that the elements $e^{3\,j}_{k\,l}$ for $k,l \in \overline{1,3}$ are the same for all matrix $E^{3\,j}$, where $j > 3$. Note that $e^{3\,j}_{3\,j} = e^{3\,j}_{j\,3} = 1$.

$$E^{i\,j}=[e^{i\,j}_{k\,l}]=\begin{bmatrix} 0 & 1 & 1 & 0 & ... & -1 & ... & -1 & ... \\ 1 & 0 & -1 & 0 & ... & ... & ... & ... & 0 \\ 1 & -1 & 0 & 0 & ... & ... & ... & ... & 0 \\ 0 & 0 & 0 & 0 & ... & ... & ... & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & ... & ... & ... & ... & 0 \\ -1 & 0 & \vdots & \vdots & ... & \ddots & \vdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & ... & ... & 0 & ... & 0 \\ -1 & \vdots & \vdots & \vdots & ... & 1 & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & ... & ... & 0 & ... & 0 \end{bmatrix}, \text{where } j>i>3.$$

Namely, $e^{i\,j}_{1\,2}=e^{i\,j}_{2\,1}=e^{i\,j}_{1\,3}=e^{i\,j}_{3\,1}=e^{i\,j}_{i\,j}=e^{i\,j}_{j\,i}=1$, $e^{i\,j}_{23}=e^{i\,j}_{32}=e^{i\,j}_{1\,min\{i,j\}}=e^{i\,j}_{1\,max\{i,j\}}=e^{i\,j}_{min\{i,j\}\,1}=e^{i\,j}_{max\{i,j\}\,1}=-1$. Otherwise, $e^{i\,j}_{k\,l}=0$.

Note that the elements $e^{i\,j}_{k\,l}$ for $k,l\in\overline{1,3}$ are the same for all matrix $E^{i\,j}$, where $j>i>3$. Note that $e^{i\,j}_{i\,j}=e^{i\,j}_{j\,i}=1$.

Denote by $\Omega=\{E^{2\,4},...,E^{2\,n};E^{3\,4},...,E^{3\,n};...;E^{n-1\,n}\}$.

Remark 19. $|\Omega|=\underbrace{1+2+..+(n-4)}_{n-4}+2(n-3)=$

$=\frac{(n-3)(n-4)}{2}+2(n-3)=\frac{n^2-3n}{2}$.

Lemma 20. *Ω is a basis in $\widetilde{Mat_s}(n,\mathbb{C})$.*

Proof. Assume there exist numbers $x_{2\,4},...,x_{2\,n};x_{3\,4},...,x_{3\,n};...;x_{n-1\,n}$ such that $\Sigma=x_{2\,4}E^{2\,4}+...+x_{2\,n}E^{2\,n}+x_{3\,4}E^{3\,4}+...+x_{3\,n}E^{3\,n}+...+x_{n-1\,n}E^{n-1\,n}=0$. Since, $\Sigma_{2\,j}=x_{2\,j}$ for $j>3$; $\Sigma_{3\,j}=x_{3\,j}$ for $j>3$; $\Sigma_{i\,j}=x_{i\,j}$ for $j>i>3$, then $x_{2\,4}=...=x_{2\,n}=x_{3\,4}=...=x_{3\,n}=...=x_{n-1\,n}=0$. It follows from the Remark 19 that $|\Omega|=dim(\widetilde{Mat_s}(n,\mathbb{C}))=\frac{n^2-3n}{2}$. Since system of vectors Ω is linearly independent and complete, then Ω is a basis in $\widetilde{Mat_s}(n,\mathbb{C})$. □

Theorem 21. *The restriction of ρ to $\widetilde{Mat_s}(n,\mathbb{C})$ is an irreducible representation.*

Proof. Using Lemma 6, we will prove that every nonzero $A\in\widetilde{Mat_s}(n,\mathbb{C})$ is cyclic for representation ρ; i.e. $Lin\{\rho(S_n)(A)\}=\widetilde{Mat_s}(n,\mathbb{C})$. Since $\widetilde{Mat_s}(3,\mathbb{C})=0$ and $\widetilde{Mat_s}(2,\mathbb{C})=0$, then let‘s prove our statement for $n>3$. It‘s sufficient to prove that $E^{2\,4}\in Lin\{\rho(S_n)(A)\}$.

Since A is nonzero, then there exists $A_{i\,j}\neq 0$ and $s\in S_n$:

$$(\rho(s)(A))_{1\,3}=(\rho((3\;j))(\rho((1\;i))(A)))_{1\,3}\neq 0. \tag{3.3}$$

If $A_{1\,3}=0$, then we replace A by $\rho(s)(A)$ and consider the matrix:

$$ {}^{1}A \;=\; A-\rho((1\;2))(A) \tag{3.4}$$

Then

$$ {}^{1}A=\begin{bmatrix} 0 & 0 & a_{1\,3}-a_{2\,3} & a_{1\,4}-a_{2\,4} & \dots & a_{1\,n}-a_{2\,n}\\ 0 & 0 & -a_{1\,3}+a_{2\,3} & -a_{1\,4}+a_{2\,4} & \dots & -a_{1\,n}+a_{2\,n}\\ a_{1\,3}-a_{2\,3} & -a_{1\,3}+a_{2\,3} & 0 & 0 & 0 & 0\\ a_{1\,4}-a_{2\,4} & -a_{1\,4}+a_{2\,4} & 0 & 0 & 0 & 0\\ \vdots & \vdots & 0 & 0 & 0 & 0\\ a_{1\,n}-a_{2\,n} & -a_{1\,n}+a_{2\,n} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.5}$$

Consider cases:

case 1.1 If $A_{i\,j}=0$, for $i,j\in\overline{3,n}$, then

$$ {}^{2}A=A, \tag{3.6}$$

Otherwise:

case 1.2 If $A_{1\,3}\neq A_{2\,3}$, then

$$ {}^{2}A=\;{}^{1}A \tag{3.7}$$

Otherwise:

case 1.3 If $A_{1\,3}=A_{2\,3}$, then

$$ {}^{2}A=\rho((1\;l))(A)-\rho((1\;2))(\rho((1\;l))(A)), \tag{3.8}$$

where $3<l\leq n$ such that $A_{1\,3}\neq A_{l\,3}$. Since $A\neq 0$ and $A\in\widetilde{Mat_s}(n,\mathbb{C})$, then there exists such $3<l\leq n$. Thus, ${}^{2}A\neq 0$.

The obtained matrix ${}^{2}A$ has $({}^{2}A)_{i\,j}=0$, for $i,j\in\overline{3,n}$ and $i,j\in\overline{1,2}$. If matrix ${}^{2}A$ is not proportional to $E^{2\,4}$, then consider cases:

case2.1. If $({}^{2}A)_{1\,3}\neq({}^{2}A)_{1\,4}$, then consider:

$$ {}^{3}A=\;{}^{2}A-\rho((3\;4))({}^{2}A). \tag{3.9}$$

Otherwise:

case2.2. If $({}^{2}A)_{1\,3}=({}^{2}A)_{1\,4}$, then consider:

$$ {}^{3}A=\rho((3\;k))({}^{2}A)-\rho((3\;4))(\rho((3\;k))({}^{2}A)). \tag{3.10}$$

Since ${}^{2}A\neq 0$ and ${}^{2}A\in\widetilde{Mat_s}(n,\mathbb{C})$, then there exists such $4<k\leq n$ that $({}^{2}A)_{1\,3}\neq({}^{2}A)_{1\,k}$. Thus, ${}^{3}A\neq 0$.

Consider matrix

$$ {}^{4}A=\frac{1}{a}\cdot({}^{3}A)=\frac{1}{a}\begin{bmatrix} 0 & 0 & a & -a & \dots & 0\\ 0 & 0 & -a & a & \dots & 0\\ a & -a & 0 & 0 & \dots & 0\\ -a & a & 0 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \text{where } ({}^{3}A)_{1\,3}=a\neq 0,$$

Note that ${}^4A = E^{2\,4}$. Every vector from Ω can be obtained by applying the representation ρ to $E^{2\,4}$ and the next equalities hold:

$E^{2\,i} = \rho((4\ i))(E^{2\,4})$, $4 < i \leq n$.

$E^{3\,i} = \rho((2\ 3))(E^{2\,i})$, $4 \leq i \leq n$.

$E^{4\,5} = \rho((3\ 4))(E^{3\,5}) + E^{2\,4}$.

$E^{4\,i} = \rho((5\ i))(E^{4\,5})$, $5 < i \leq n$.

$E^{k\,i} = \rho((4\ k))(E^{4\,i})$, $4 < k < n$, $k+1 \leq i \leq n$.

Since $E^{i\,j} \in Lin\{\rho(G)v\}$ for each $E^{i\,j} \in \Omega$, for each $v \in \widetilde{Mat_s}(n, \mathbb{C})$ and Ω is a basis, then for every $w \in \widetilde{Mat_s}(n, \mathbb{C})$, $w \in Lin\{\rho(G)v\}$. Thus, the restriction of ρ to $\widetilde{Mat_s}(n, \mathbb{C})$ is an irreducible representation. $\square$

3.3 Irreducible components of $Mat_{as}(n, \mathbb{C})$

Let's consider the vector system $\{e_i^{as}\}_{i=1}^n$ (where n is a size of the matrix) consisting of vectors of the form:

$$e_i^{as} = [\alpha_{rc}] = \begin{bmatrix} 0 & ... & 0 & 1 & 0 & ... & 0 \\ 0 & ... & 0 & 1 & 0 & ... & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & ... & -1 & 0 & -1 & ... & -1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & ... & 0 & 1 & 0 & ... & 0 \end{bmatrix}.$$

Namely, elements of matrix $\alpha_{rc} = -1$ for such indexes r and c that $r = i, c \in \overline{1,n}\backslash\{i\}$, $\alpha_{rc} = 1$ for such indexes r and c that $r \in \overline{1,n}\backslash\{i\}, c = i$ and $\alpha_{rc} = 0$ otherwise.

Lemma 22. *$\{e_i^{as}\}_{i=2}^n$ is a linearly independent vector system. $Lin\{\{e_i^{as}\}_{i=1}^n\}$ is ρ invariant.*

Proof. Let's assume there exist such numbers $\lambda_1, \lambda_2, ..., \lambda_{n-1}$ that $\Sigma_{i=2}^n \lambda_{i-1} e_i^{as} = 0$. The next equality holds:

$$\sum_{i=2}^n \lambda_i e_i^{as} = \begin{bmatrix} 0 & \lambda_2 & \lambda_3 & ... & \lambda_n \\ -\lambda_2 & 0 & -\lambda_2 + \lambda_3 & ... & -\lambda_2 + \lambda_n \\ -\lambda_3 & \lambda_2 - \lambda_3 & 0 & ... & -\lambda_3 + \lambda_n \\ \vdots & \vdots & \vdots & \ddots & \\ -\lambda_n & \lambda_2 - \lambda_n & \lambda_3 - \lambda_n & ... & 0 \end{bmatrix} = 0. \quad (3.11)$$

It follows from the equality 3.11 that $\lambda_2 = \lambda_3 = ... = \lambda_n = 0$. So $\{e_i^{as}\}_{i=2}^n$ are linearly independent vectors.

Note that $e_2^{as} + e_3^{as} + ... + e_n^{as} = -e_1^{as}$. For any $i \in \overline{1,n}$, $\rho(e_i^{as}) \in Lin\{\{e_i^{as}\}_{i=2}^n\}$. So for any $\rho(x_2 e_2^{as} + ... + x_n e_n^{as}) = \widetilde{x_2} e_2^{as} + ... + \widetilde{x_n} e_n^{as} \in Lin\{\{e_i^{as}\}_{i=2}^n\}$, where $x_2, ..., x_n \in \mathbb{C}$ $\square$

Let's denote $X_{as} = Lin\{\{e_i^{as}\}_{i=2}^n\}$

Lemma 23. *X_{as} is an irreducible representation.*

Proof. Let's take an arbitrary nonzero $x \in X_{as}$, then x can be represented as $x = x_1 e_2^{as} + ... + x_{n-1} e_n^{as}$, where $x_1, ..., x_{n-1} \in \mathbb{C}$.

If $x_1 = x_2 = ... = x_{n-1}$, then $\rho((1\ 2))(x) = -x_1 e_2^{as}$.

If $x_1 \neq x_2$, then ${}^1x = \frac{1}{x_1 - x_2}(x - \rho((2\ 3))(x))$.

If $x_1 = x_2 \neq 0$ and there exists such $2 < l \leq n$ that $x_1 \neq x_l$, then consider ${}^1x = \frac{1}{\hat{x}_1 - \hat{x}_2}(\hat{x} - \rho((2\ 3))(\hat{x}))$, where $\hat{x} = \rho((2\ l))(x)$.

If $x_1 = x_2 = 0$, then consider ${}^1x = \frac{1}{\hat{x}_1 - \hat{x}_2}(\hat{x} - \rho((2\ 3))(\hat{x}))$, where $\hat{x} = \rho((2\ l))(x)$ and $2 < l \leq (n-1)$ such that $x_l \neq 0$.

Note that ${}^1x = e_2^{as} - e_3^{as}$.

$$
{}^1x = \begin{bmatrix}
0 & 1 & -1 & 0 & 0 & ... & 0 \\
-1 & 0 & -2 & -1 & -1 & ... & -1 \\
1 & 2 & 0 & 1 & 1 & ... & 1 \\
0 & 1 & -1 & 0 & 0 & ... & 0 \\
0 & 1 & -1 & 0 & 0 & ... & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 1 & -1 & 0 & 0 & ... & 0
\end{bmatrix}.
$$

Let's show that $e_2^{as} \in Lin\{\rho(G)({}^1x)\}$:

${}^2x = {}^1x + \rho((2\ 4))({}^1x) + ... + \rho((2\ i))({}^1x) + .. + \rho((2\ n))({}^1x)$

$$
{}^2x = \begin{bmatrix}
0 & 1 & -(n-2) & 1 & 1 & ... & 1 \\
-1 & 0 & -(n-1) & 0 & 0 & ... & 0 \\
n-2 & n-1 & 0 & n-1 & n-1 & ... & n-1 \\
-1 & 0 & -(n-1) & 0 & 0 & ... & 0 \\
-1 & 0 & -(n-1) & 0 & 0 & ... & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & 0 & -(n-1) & 0 & 0 & ... & 0
\end{bmatrix}.
$$

${}^3x = {}^2x + \rho((1\ 3))({}^2x)$.

$$
{}^3x = \begin{bmatrix}
0 & n & 0 & n & n & ... & n \\
-n & 0 & -n & 0 & 0 & ... & 0 \\
0 & n & 0 & n & n & ... & n \\
-n & 0 & -n & 0 & 0 & ... & 0 \\
-n & 0 & -n & 0 & 0 & ... & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-n & 0 & -n & 0 & 0 & ... & 0
\end{bmatrix}.
$$

${}^4x = {}^3x + n\rho((3\ 4))({}^1x) + ... + n\rho((3\ n))({}^1x)$.

${}^4x$ is proportional to e_2^{as}. Namely, ${}^4x = n(n-2)e_2^{as}$.

For any $0 < i \leq n$ the next equality holds: $e_i^{as} = \rho((2\ i))(e_2^{as})$.

Thus, $Lin\{\rho(G)a\} = X_{as}$, for each $a \in X_{as}$. Hence, the restriction of ρ to X_{as} is an irreducible subrepresentation in $\widetilde{Mat}_{as}(n, \mathbb{C})$. $\square$

Let s denote by $\widetilde{Mat}_{as}(n, \mathbb{C})$ the subspace of antisymmetric matrices $[m_{ij}] \subset Mat_{as}(n, \mathbb{C})$ such that $\Sigma_{j=1}^{n} m_{ij} = 0$ for all $i \in \overline{1, n}$.

Proposition 24. *The subspace $Mat_{as}(n, \mathbb{C})$ is an orthogonal sum of $X_{as} \oplus \widetilde{Mat}_{as}(n, \mathbb{C})$.*

Proof. Verification shows that for any $A \in \widetilde{Mat}_{as}(n, \mathbb{C})$ and for any $B \in X_{as}$ $\langle A, B \rangle = 0$.

Suppose $0 \neq v \in Mat(n, \mathbb{C})$ such that v is orthogonal to X_{as} and $\widetilde{Mat}_{as}(n, \mathbb{C})$. Since v is orthogonal to X_{as}, then v is orthogonal to each basis vector in X_{as}. Thus $\Sigma_{j=1}^{n} v_{i\,j} = 0$ for each $i \in \overline{1, n}$. Hence $v \in \widetilde{Mat}_{as}(n, \mathbb{C})$. Since v is orthogonal to $\widetilde{Mat}_{as}(n, \mathbb{C})$, then $v = 0$. Hence, $Mat_{as}(n, \mathbb{C}) = X_{as} \oplus \widetilde{Mat}_{as}(n, \mathbb{C})$.

$\square$

Remark 25.

$$dim(\widetilde{Mat}_{as}(n, \mathbb{C})) = \frac{n^2 - n}{2} - (n - 1) = \frac{n^2 - 3n + 2}{2}.$$

Let's consider the next vector system in $\widetilde{Mat}_{as}(n, \mathbb{C})$:

$$E_{ij} = [(E_{ij})_{k\,l}] = \begin{bmatrix} 0 & \dots & 1 & \dots & -1 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -1 & \dots & 0 & \dots & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \dots & -1 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix}$$

where $(E_{ij})_{ij} = (E_{ij})_{j1} = (E_{ij})_{1i} = 1, (E_{ij})_{ji} = (E_{ij})_{1j} = (E_{ij})_{i1} = -1$, where $i = \overline{2, n-1}$, $j = \overline{i+1, n}$ for $n > 2$, otherwise $e_{ij} = 0$.

Lemma 26. *E_{ij} is a basis in $\widetilde{Mat}_{as}(n, \mathbb{C})$.*

Proof. $dim(Lin\{E_{ij}\}) = \frac{n^2-3n+2}{2} = dim(\widetilde{Mat}_{as}(n, \mathbb{C}))$. Suppose that there exist such $\lambda_{2\,3}, \lambda_{2\,4}, \dots, \lambda_{n-1\,n}$ that $\Sigma = \lambda_{2\,3}E_{2\,3} + \dots + \lambda_{n-1\,n}E_{n-1\,n} = 0$. Since $\Sigma_{i\,j} = x_{i\,j}$, for $i = \overline{2, n-1}$, $j = \overline{i+1, n}$, then $\lambda_{2\,3} = \lambda_{2\,4} = \dots = \lambda_{n-1\,n} = 0$. Since the vector system is linearly independent and satisfies the spanning property, the vector system $\{E_{ij}\}$ is a basis in $\widetilde{Mat}_{as}(n, \mathbb{C})$. $\square$

Theorem 27. *The restriction of ρ to $\widetilde{Mat}_{as}(n,\mathbb{C})$ is an irreducible representation.*

Proof. Let's take an arbitrary nonzero $A = [A_{i\,j}] \in \widetilde{Mat}_{as}(n,\mathbb{C})$.

Since A is nonzero, then there exists $A_{i\,j} \neq 0$ and $s \in S_n$:

$$(\rho(s)(A))_{1\,3} = (\rho((3\;j))(\rho((1\;i))(A)))_{1\,3} \neq 0. \tag{3.12}$$

If $A_{1\,3} = 0$, then we replace A by $\rho(s)(A)$ and consider the matrix:

$$^1A \;=\; A - \rho((1\;2))(A) \tag{3.13}$$

Consider cases:

case1.1. If $A_{i\,j} = 0$, for any $i, j \in \overline{3,n}$, then

$$^2A = A, \tag{3.14}$$

otherwise:

case1.2. If $A_{1\,3} \neq A_{2\,3}$, then

$$^2A = \;^1A, \tag{3.15}$$

otherwise:

case1.3. If $A_{1\,3} = A_{2\,3}$ and there are nonzero elements in the third column (there exist $i \in \overline{1,n}$ such that $A_{i\,3} \neq 0$), then

$$^2A = \rho((1\;l))(A) - \rho((1\;2))(\rho((1\;l))(A)), \tag{3.16}$$

where $l \in \mathbb{N}$ such that $A_{1\,3} \neq A_{l\,3}$. Since $A \neq 0$ and $A \in \widetilde{Mat}_{as}(n,\mathbb{C})$, then there exists such $3 < l \leq n$.

Let ^2A is a matrix obtained after all previous steps that is not represented as $aE_{2\,3} + bE_{2\,4}$, then consider cases:

case2.1. If $(^2A)_{1\,3} \neq (^2A)_{1\,4}$, then consider:

$$^3A = \;^2A - \rho((3,4))(^2A). \tag{3.17}$$

Otherwise:

case2.2. If $(^2A)_{1\,3} = (^2A)_{1\,4}$ and there exists such $4 < k \leq n$ that $(^2A)_{1\,3} \neq (^2A)_{1\,k}$, then consider:

$$^3A = \rho((3\;k))(^2A) - \rho((3\;4))(\rho((3\;k))(^2A)). \tag{3.18}$$

Otherwise:

case2.3.

$$^2A = \sigma \begin{bmatrix} 0 & -(n-2) & 1 & 1 & 1 & \cdots & 1 \\ n-2 & 0 & -1 & -1 & -1 & \cdots & -1 \\ -1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{3.19}$$

where the number $\sigma \neq 0$.

Now we note that

$$ {}^{3,1}A = \rho((2\ \ n))\left({}^{2}A\right) = \sigma \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & \cdots & -(n-2) \\ -1 & 0 & 0 & 0 & 0 & \cdots & 1 \\ -1 & 0 & 0 & 0 & 0 & \cdots & 1 \\ -1 & 0 & 0 & 0 & 0 & \cdots & 1 \\ -1 & 0 & 0 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (n-2) & -1 & -1 & -1 & -1 & \cdots & 0 \end{bmatrix} \quad \text{and} $$

$$ \rho((1\ \ 2))\left({}^{3,1}A\right) = \sigma \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 1 & 1 & 1 & \cdots & -(n-2) \\ 0 & -1 & 0 & 0 & 0 & \cdots & 1 \\ 0 & -1 & 0 & 0 & 0 & \cdots & 1 \\ 0 & -1 & 0 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & (n-2) & -1 & -1 & -1 & \cdots & 0 \end{bmatrix} $$

Hence

$$ {}^{3,2}A = {}^{3,1}A - \rho((1\ \ 2))\left({}^{3.1}A\right) \tag{3.20} $$

$$ {}^{3,2}A = \sigma \begin{bmatrix} 0 & 2 & 1 & 1 & 1 & \cdots & -(n-1) \\ -2 & 0 & -1 & -1 & -1 & \cdots & (n-1) \\ -1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (n-1) & -(n-1) & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{3.21} $$

It follows from (3.19) and (3.21) that the matrix $B = {}^{2}A + \frac{1}{n-1}\,{}^{3,2}A$ satisfies the conditions

$$ (B)_{nk} = (B)_{kn} \quad \text{for all} \quad k = 1, 2, \ldots n \quad \text{and} \quad B \neq 0 \quad \text{for} \quad n \geq 3. \tag{3.22} $$

Applying mathematical induction on n, we obtain that

$$ {}^{3}A = aE_{2\,3} + bE_{2\,4} \in Lin\{\rho(S_n)({}^{2}A)\}, \text{ where } a \neq 0 \text{ or } b \neq 0. \tag{3.23} $$

The matrix ${}^{3}A$ obtained after **case2** has the next form:

$$ {}^{3}A = aE_{2\,3} + bE_{2\,4}. \tag{3.24} $$

Let's show that ${}^3A = aE_{2\,3} + bE_{2\,4} \in Lin\{\rho(S_n)(A)\}$ for $a, b \neq 0$:

$$
{}^3A = aE_{2\,3} + bE_{2\,4} = -\begin{bmatrix} 0 & -a-b & a & b & 0 & \dots & 0 \\ a+b & 0 & -a & -b & 0 & \dots & 0 \\ -a & a & 0 & 0 & 0 & \dots & 0 \\ -b & b & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}
$$

let's perform the following calculations:

$$
{}^3A + (1\ 4){}^3A = -\begin{bmatrix} 0 & -a & a & 0 & \dots & 0 \\ a & 0 & -2a & a & \dots & 0 \\ -a & 2a & 0 & -a & \dots & 0 \\ 0 & -a & -a & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix} = -a\begin{bmatrix} 0 & -1 & 1 & 0 & \dots & 0 \\ 1 & 0 & -2 & 1 & \dots & 0 \\ -1 & 2 & 0 & -1 & \dots & 0 \\ 0 & -1 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}
$$

$$
B_1 = \rho((1\ 3))(\frac{1}{a}({}^3A + \rho((1\ 4))({}^3A))) \tag{3.25}
$$

$B1$ has the form:

$$
B_1 = \begin{bmatrix} 0 & -2 & 1 & 1 & \dots & 0 \\ 2 & 0 & -1 & -1 & \dots & 0 \\ -1 & 1 & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}
$$

Let's perform the following sequence of calculations with the matrix B_1:
$B_2 = B_1 + (1\ 4)B_1$
$B_{3,1} = \rho((2\ 3))(B_2))$
$B_{3,2} = \rho((3\ 4))(B_{3,1}))$
$B_{3,3} = \rho((2\ 3))(B_{3,2}))$
$B_4 = B_2 + 2B_{3,3}$
$B_5 = \rho((3\ 4))(B_4)$
$B_{6,1} = \rho((2\ 3))(B_5))$
$B_{6,2} = \rho((3\ 4))(B_{6,1}))$
$B_{6,3} = \rho((2\ 3))(B_{6,2}))$
$B_7 = B_5 + 3B_{6,3}$

$$B_8 = \frac{1}{8}B_7 \begin{bmatrix} 0 & 1 & -1 & 0 & \dots & 0 \\ -1 & 0 & 1 & 0 & \dots & 0 \\ 1 & -1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix} = E_{2\,3}$$

$E_{2\,i} = \rho((3\ i))(E_{2\,3}))$
$E_{k\,i} = \rho((k\ 2))(E_{2\,i}))$, where $(k > i)$.

Hence, $Lin\{\rho(E_{2\,3})\} = \widetilde{Mat}_{as}(n, \mathbb{C})$. Thus, restriction of ρ to $\widetilde{Mat}_{as}(n, \mathbb{C})$ is an irreducible representation.

□

4 Action of S_∞ on the space of Hilbert-Schmidt operators by conjugation.

Definition 28. A unitary representation of an abstract group G in a complex Hilbert space is a homomorphism ρ: $G \to U(H)$, where $U(H)$ is a group of unitary operators on H.

Let‘s consider S_∞ as the group of all finite permutations of the set $\mathbb{N}$, where $\pi : \mathbb{N} \to \mathbb{N}$ such that $s(j) \neq j$ for finitely many elements of $\mathbb{N}$.

Definition 29. Hilbert-Schmidt operator (HS-operator) is a continuous linear map $A : H \to H$, that acts on a Hilbert space H and has finite Hilbert-Schmidt norm:

$$\|A\|^2_{HS} := \sum_{i=1}^{\infty} \|Ae_i\|^2_H = \sum_{i,j}^{\infty} |\langle e_i, Ae_j\rangle|^2 \tag{4.26}$$

where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis.

Definition 30. Let H be a separable Hilbert space, $\{e_k\}_{k=1}^{\infty}$ an orthonormal basis and $A : H \to H$ a positive bounded linear operator on H. The trace of A is denoted by $Tr(A)$ and defined as:

$$Tr(A) = \sum_{k=1}^{\infty} \langle Ae_k, e_k\rangle, \tag{4.27}$$

Remark 31. The set of all HS-operators on H forms a Hilbert space, the inner product can be defined as: $\langle A, B\rangle_{HS} = Tr(B^*A) = \Sigma_i \langle Ae_i, Be_i\rangle$.

Let‘s define the norm of HS operator A by

$$\|A\|_{HS} = \sqrt{\langle A, A\rangle} = \sqrt{Tr(AA^*)} < \infty \tag{4.28}$$

Every HS-operator T is bounded. Let $T_{i\,j} = \langle Te_j, e_i\rangle$.

Let's define the embedding of S_∞ into unitary group $U(H)$ by:

$$S_\infty \ni \pi \to [\delta_{i\,s(j)}] \in U(H), \tag{4.29}$$

where $\delta_{i\,s(j)}$ is a Kronecker delta. Let's define the representation ρ by:

$$\rho(\pi)(T) = [\delta_{i\,\pi(j)}]\,T\,[\delta_{i\,\pi(j)}]^{-1}, \tag{4.30}$$

where T is a Hilbert-Schmidt operator.

Remark 32. Let $A, B \in HS$ then $Tr(AB) = Tr(BA)$.

Statement 1. *A representation ρ is unitary.*

Proof. Let $A, B \in HS$, then

$$\begin{aligned} \langle \rho(\pi)(A), B\rangle = Tr(B^*\rho(\pi)(A)) = Tr(B^*[\delta_{i\,\pi(j)}]\;A\;[\delta_{i\,\pi(j)}]^{-1}) \\ = Tr([\delta_{i\,\pi(j)}]^{-1}\;B^*\;[\delta_{i\,\pi(j)}]A) = Tr(([\delta_{i\,\pi(j)}]^{-1}\;B\;[\delta_{i\,\pi(j)}])^*A) \\ = \langle A, \rho(\pi)^*(B)\rangle = \langle A, \rho(\pi)^{-1}(B)\rangle \end{aligned}$$

Hence $\rho^*(\pi)(T) = [\delta_{i\,\pi(j)}]^{-1}T[\delta_{i\,\pi(j)}]$. Thus,

$$\rho^*(\pi)(\rho(\pi)(T)) = \rho(\pi)(\rho^*(\pi)(T)) = id. \tag{4.31}$$

□

We suppose that basis $\{e_k\}_{k=1}^{\infty}$ is fixed and we identify operator with a corresponding matrix.

Definition 33. Let $A = [A_{m\,n}]$. Then $(A^T)_{m\,n} = A_{n\,m}$.

Proposition 34. *A is a HS-matrix, then A^T is a HS-matrix.*

Proof. Proof follows from the Definition 29. □

Proposition 35. *Every HS-matrix A can be represented as a sum of $\frac{A+A^T}{2}$ and $\frac{A-A^T}{2}$.*

Denote the space of Hilbert-Schmidt matrixes by HS.
Define the next subspaces:
$Mat_s(HS) = \{A \in HS : A = A^T\}$,
$Mat_{as}(HS) = \{A \in HS : A = -A^T\}$, where A^T is an operator which matrix is a transposed matrix of A matrix.

Remark 36. A matrix $\frac{A+A^T}{2}$ is symmetric and a matrix $\frac{A-A^T}{2}$ is anti-symmetric.

Definition 37. A bounded linear operator $T : H \to H$ is called trace class if $Tr(|T|) < \infty$, where $|T| = \sqrt{T^*T}$ is a non-negative square root.

Proposition 38. *Product of two Hilbert-Schmidt matrixes is a trace class matrix.*

Proposition 39. *Every trace class operator is equal to the composition of two Hilbert-Schmidt operators.*

Proposition 40. *Let A is a trace class operator, then A^T is also a trace class operator.*

Lemma 41. *$Mat_s(HS)$ and $Mat_{as}(HS)$ are mutually orthogonal.*

Proof. Let's take $A \in Mat_s(HS)$ and $B \in Mat_{as}(HS)$. Then $B^* \in Mat_{as}(HS)$.

$$\langle A, B\rangle = Tr(B^*A) = Tr((B^*A)^T) = Tr(A^T B^{*T}) = \\ = -Tr(AB^*) = -Tr(B^*A) = -\langle A, B\rangle.$$

Thus, $Tr(B^*A) = \langle A, B\rangle = 0$. □

Lemma 42. *$Mat_s(HS)$ and $Mat_{as}(HS)$ are ρ-invariant.*

Proof. Let $m_1 \in Mat_s(HS)$. By definition of ρ, we have

$$\rho(s)(m_1) = [\delta_{i\ s(j)}] \cdot m_1 \cdot [\delta_{i\ s(j)}]^{-1}, and$$

$$(\rho(s)(m_1))^t = ([\delta_{i\ s(j)}]^{-1})^t \cdot m_1^t \cdot [\delta_{i\ s(j)}]^t = [\delta_{i\ s(j)}] \cdot m_1 \cdot [\delta_{i\ s(j)}]^{-1}.$$

Let $m_2 \in Mat_{as}(HS)$. By definition of ρ, we have

$$\rho(s)(m_2) = [\delta_{i\ s(j)}] \cdot m_2 \cdot [\delta_{i\ s(j)}]^{-1}, and$$

$$(\rho(s)(m_2))^t = ([\delta_{i\ s(j)}]^{-1})^t \cdot m_2^t \cdot [\delta_{i\ s(j)}]^t = -[\delta_{i\ s(j)}] \cdot m_2 \cdot [\delta_{i\ s(j)}]^{-1}.$$

□

4.1 Irreducible components of $Mat_s(HS)$.

Let's denote the subspace

$$D_0 = \{A \in Mat_s(HS) : a_{ij} = 0 \text{ for } i \neq j, \sum_{i=1}^{\infty} a_{ii} = 0\}. \quad (4.32)$$

$$diag(HS) = \{A \in Mat_s(HS) : a_{ij} = 0 \text{ for } i \neq j\}. \quad (4.33)$$

Let's denote by $\widetilde{Mat}_s(HS)$ the subspace of symmetric HS operators $[m_{ij}]$ such that $m_{ii} = 0$ and $\sum_{j=1}^{\infty} m_{ij} = 0$ for all $i \in \mathbb{N}$.

Let's denote by D_0^{fin} such elements from D_0 that only a finite number of elements of the matrix are nonzero.

Let's denote by $\widetilde{Mat}_s^{fin}(HS)$ such elements from $\widetilde{Mat}_s(HS)$ that only a finite number of elements of the matrix are nonzero.

Definition 43. The closed subspace $\acute{H} \subset H$ is called $\rho(G)$-invariant if $\rho(G)\acute{H} \subset \acute{H}$.

Let‘s denote the smallest invariant subspace in H containing $\rho(G)v$ by $\overline{Lin\{\rho(G)v\}}$.

Definition 44. Representation ρ is irreducible if and only if for any nonzero $v \in H$ the equality holds:

$$H = \overline{Lin\{\rho(G)v\}} \tag{4.34}$$

Proposition 45. *A unitary representation ρ is irreducible if and only if its commutator consists of only of scalar operators.*

Let‘s show that the restriction of ρ to $diag(HS)$ is an irreducible representation.

Lemma 46. *D_0 is a dense set in $diag(HS)$.*

Proof. Let $x = diag(x_1, ..., x_n, ...) \in diag(HS)$. Fix $\epsilon > 0$. Then there exists $N_\epsilon \in \mathbb{N}$ such that

$$\|x - x(N_\epsilon)\|_{HS} < \epsilon/2, \tag{4.35}$$

where $x(N_\epsilon) = (x_1, ..., x_{N_\epsilon}, 0, ...)$.
Put

$$x(N_\epsilon, k) = (x_1, ..., x_{N_\epsilon}, \underbrace{\frac{S}{k}, ..., \frac{S}{k}}_{\text{k times}}, 0, ...), \text{where } S = -\sum_{i=1}^{N_\epsilon} x_i. \tag{4.36}$$

By definition $x(N_\epsilon, k) \in D_0$.
It follows from 4.35 and 4.36 that

$$\|x(N_\epsilon) - x(N_\epsilon, k)\|_{HS} = \sqrt{\sum_{i=n+1}^{n+k} \frac{|S|^2}{k}} = \sqrt{k\frac{|S|^2}{k}} = \frac{|S|}{\sqrt{k}}. \tag{4.37}$$

It‘s clear that for $k > \frac{4|S|^2}{\epsilon^2}$ we have

$$\|x(N_\epsilon) - x(N_\epsilon, k)\|_{HS} < \epsilon/2. \tag{4.38}$$

Thus, by 4.35, we obtain

$$\|x - x(N_\epsilon, k)\| \leq \|x - x(N_\epsilon)\| + \|x(N_\epsilon) - x(N_\epsilon, k)\| < \epsilon/2 + \epsilon/2 = \epsilon.$$

$\square$

Remark 47. $\overline{D_0^{fin}} = \overline{D_0} = diag(HS)$.

Theorem 48. *The restriction of ρ to $diag(HS)$ is an irreducible representation.*

Proof. Let's take an arbitrary nonzero $a \in diag(HS)$, $a = diag(a_1, a_2, ...,)$. If $(k\ l)$ is an arbitrary transposition, then there exists $s \in S_n$, such that

$$(1\ 2) = s \cdot (k\ l) \cdot s^{-1} \tag{4.39}$$

There exists $(k\ l)$ such that $a_k \neq a_l$. Then

$$^1a = a - \rho((k\ l))(a) \neq 0. \tag{4.40}$$

Thus, we have from 4.40 that

$$\begin{aligned} 0 \neq\ ^1a = a - \rho(s^{-1} \cdot (1\ 2) \cdot s)(a) = \\ = a - \rho(s^{-1})(\rho((1\ 2))(\rho(s)(a))). \end{aligned}$$

It follows from this:

$$\begin{aligned} \rho(s)(^1a) = \rho(s)(a) - \rho(s)(\rho(s^{-1})(\rho((1\ 2))(\rho(s)(a)))) = \\ = \rho(s)(a) - \rho((1\ 2))(\rho(s)(a)) \neq 0. \end{aligned}$$

If $a - \rho((1\ 2))(a) = 0$, then we replace a by $\rho(s)(a)$. Thus, without lose of generality, we assume

$$b = a - \rho((1\ 2))(a) = \lambda \cdot diag(1, -1, 0, ..., 0), \text{where } \lambda \neq 0.$$

By Remark 47, we have that $\overline{Lin\{\rho(G)b\}} = diag(HS)$. Thus, $diag(HS)$ is an irreducible representation. □

Let's show that the restriction of ρ to $Mat_s(HS)$ is an irreducible representation.

Lemma 49. *$\widetilde{Mat}_s(HS)$ is a dense set in $Mat_s(HS)$.*

Proof. Fix $\epsilon > 0$.
For given $x \in Mat_s(HS)$, find $N \in \mathbb{N}$ such that $n > N$:

$$\|x - x(n)\| < \epsilon/2, \tag{4.41}$$

where

$$x(N) = \begin{bmatrix} 0 & x_{1\,2} & x_{1\,3} & \dots & x_{1\,N-1} & x_{1\,N} & 0 & \dots \\ x_{1\,2} & 0 & x_{2\,3} & \dots & x_{2\,N-1} & x_{2\,N} & 0 & \dots \\ x_{1\,3} & x_{2\,3} & 0 & \dots & x_{3\,N-1} & x_{3\,N} & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ x_{1\,N-1} & x_{2\,N-1} & x_{3\,N-1} & \dots & 0 & x_{N-1\,N} & 0 & \dots \\ x_{1\,N} & x_{2\,N} & x_{3\,N} & \dots & x_{N-1\,N} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots \end{bmatrix}.$$

Namely, $x(N)_{i\,j} = x_{i\,j}$, for any $i \leq n$, $j \leq n$; $x(N)_{i\,j} = 0$, otherwise.

Let‘s introduce

$$S_i = \sum_{j=1}^{N} x(N)_{i\,j} \tag{4.42}$$

$$\Sigma = \sum_{i=1}^{N} S_i = \sum_{i=1}^{N}\sum_{j=1}^{N} x_{i\,j}. \tag{4.43}$$

$$x(N,k) = x(N) + {}^{0}x(N,k), \text{where } x(N,k) \in \widetilde{Mat}_s(HS), \tag{4.44}$$

$$ {}^{0}x(N,k) = \begin{bmatrix} 0 & ... & 0 & | & \frac{-S_1}{k} & \frac{-S_1}{k} & ... & \frac{-S_1}{k} & 0 & ... \\ \vdots & \ddots & \vdots & | & \frac{-S_2}{k} & \frac{-S_2}{k} & ... & \frac{-S_2}{k} & 0 & ... \\ 0 & ... & 0 & | & \vdots & \vdots & ... & \vdots & \vdots & \vdots \\ \_ & \_ & \_ & | & \frac{-S_N}{k} & \frac{-S_N}{k} & ... & \frac{-S_N}{k} & 0 & ... \\ -\frac{S_1}{k} & -\frac{S_2}{k} & ... & -\frac{S_N}{k} & 0 & \frac{\Sigma}{k(k-1)} & ... & \frac{\Sigma}{k(k-1)} & 0 & ... \\ -\frac{S_1}{k} & -\frac{S_2}{k} & ... & -\frac{S_N}{k} & \frac{\Sigma}{k(k-1)} & 0 & ... & \frac{\Sigma}{k(k-1)} & 0 & ... \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -\frac{S_1}{k} & -\frac{S_2}{k} & ... & -\frac{S_N}{k} & \frac{\Sigma}{k(k-1)} & \frac{\Sigma}{k(k-1)} & ... & 0 & 0 & ... \\ 0 & 0 & ... & 0 & 0 & 0 & ... & 0 & 0 & ... \\ \vdots & \vdots & ... & \vdots & \vdots & \vdots & ... & \vdots & \vdots & \ddots \end{bmatrix}.$$

Namely,

$$\begin{aligned} &{}^{0}x(N,k)_{i\,j} = 0, \text{ if } i \le N \text{ and } j \le N; \\ &{}^{0}x(N,k)_{i\,j} = \frac{-S_i}{k} \text{ if } i \le N \text{ and } N < j \le N+k; \\ &{}^{0}x(N,k)_{i\,j} = \frac{-S_i}{k} \text{ if } j \le N \text{ and } N < i \le N+k; \\ &{}^{0}x(N,k)_{i\,j} = \frac{\Sigma}{k(k-1)} \text{ if } N < j \le N+k \text{ and } N < i \le N+k \text{ and } i \neq j. \\ &x(N)_{i\,j} = 0, \text{ otherwise.} \end{aligned} \tag{4.45}$$

It follows from 4.56 that

$$\|x(N) - x(N,k)\|_{HS} = \sqrt{\frac{2}{k}|S_1|^2 + ... + \frac{2}{k}|S_n|^2 + \frac{1}{k(k-1)}|\Sigma|^2}.$$

Note that there exists k such that

$$\|x(N) - x(N,k)\|_{HS} < \epsilon/2. \tag{4.46}$$

Thus, by 4.53 and 4.57, we obtain

$$\|x - x(N,k)\| \le \|x - x(N)\| + \|x(N) - x(N,k)\| < \epsilon/2 + \epsilon/2 = \epsilon$$

□

Remark 50. $\overline{\widetilde{Mat}_s^{fin}(HS)} = \overline{\widetilde{Mat}_s(HS)} = Mat_s(HS)$.

Theorem 51. *The restriction of ρ to $Mat_s(HS)$ is an irreducible representation.*

Proof. Let's take an arbitrary nonzero $A = [A_{i\,j}] \in Mat_s(HS)$. Since A is nonzero, then there exists $A_{i\,j} \neq 0$ and $s \in S_n$:

$$(\rho(s)(A))_{1\,3} = (\rho((3\ j))(\rho((1\ i))(A)))_{1\,3} \neq 0. \tag{4.47}$$

If $A_{1\,3} = 0$, then we replace A by $\rho(s)(A)$ and consider the matrix:

$$^1A \ = \ A - \rho((1\ 2))(A) \tag{4.48}$$

Consider cases:

case 1.1. $A_{1\,3} \neq A_{2\,3}$, then

$$^2A = \ ^1A. \tag{4.49}$$

Otherwise:

case 1.2. If $A_{1\,3} = A_{2\,3}$, then consider:

$$^2A = \rho((1\ l))(A) - \rho((1\ 2))(\rho((1\ l))(A)), \tag{4.50}$$

where $l \in \mathbb{N}$ such that $A_{1\,3} \neq A_{l\,3}$. Since $A \neq 0$ and $A \in \widetilde{Mat}_s(HS)$, then there exists such $l > 3$ that $A_{1\,3} \neq A_{l\,3}$.

Let ^2A is a matrix obtained after previous steps, then consider cases:

case 2.1. If $(^2A)_{1\,3} \neq (^2A)_{1\,4}$, then consider:

$$^3A = \ ^2A - \rho((3\ 4))(^2A). \tag{4.51}$$

Otherwise:

case 2.2. If $(^2A)_{1\,3} = (^2A)_{1\,4}$, then consider

$$^3A = \ \rho((3\ k))(^2A) - \rho((3\ 4))(\rho((3\ k))(^2A)). \tag{4.52}$$

Since $^2A \neq 0$ and $^2A \in \widetilde{Mat}_s(HS)$, then there exists such $k > 4$ that $(^2A)_{1\,3} \neq (^2A)_{1\,k}$. Thus, we obtain:

$$^3A = \lambda \begin{bmatrix} 0 & 0 & 1 & -1 & 0 & \dots \\ 0 & 0 & -1 & 1 & 0 & \dots \\ 1 & -1 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Using Th 21, we obtain that for every element $v \in \widetilde{Mat}_s^{fin}(HS)$, it follows that $v \in Lin\{\rho(G)(A)\}$. Thus, $\widetilde{Mat}_s^{fin}(HS) \subset Lin\{\rho(G)(A)\}$. Using Remark 50 and ρ-invariance of $Mat_s(HS)$, we obtain that $Mat_s(HS) = \overline{\widetilde{Mat}_s(HS)} = \overline{\widetilde{Mat}_s^{fin}(HS)} = \overline{Lin\{\rho(G)(A)\}}$. Thus, the restriction of ρ to $Mat_s(HS)$ is an irreducible representation.

□

4.2 Irreducible components of $Mat_{as}(HS)$.

Let's denote by $\widetilde{Mat}_{as}(HS)$ the subspace of antisymmetric HS operators $[m_{ij}]$ such that $m_{ii} = 0$ and $\Sigma_{j=1}^{\infty} m_{ij} = 0$ for all $i \in \mathbb{N}$.

Let's denote by $\widetilde{Mat}_{as}^{fin}(HS)$ the subspace of elements from $\widetilde{Mat}_{as}(HS)$ such that only a finite number of elements of the matrix are nonzero.

Lemma 52. *$\widetilde{Mat}_{as}(HS)$ is a dense set in $Mat_{as}(HS)$.*

Proof. Fix $\epsilon > 0$.
For given $x \in Mat_{as}(HS)$, find $N \in \mathbb{N}$ such that $n > N$:

$$\|x - x(n)\| < \epsilon/2, \tag{4.53}$$

where

$$x(N) = \begin{bmatrix} 0 & x_{1\,2} & x_{1\,3} & \dots & x_{1\,N-1} & x_{1\,N} & 0 & \dots \\ -x_{1\,2} & 0 & x_{2\,3} & \dots & x_{2\,N-1} & x_{2\,N} & 0 & \dots \\ -x_{1\,3} & -x_{2\,3} & 0 & \dots & x_{3\,N-1} & x_{3\,N} & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ -x_{1\,N-1} & -x_{2\,N-1} & -x_{3\,N-1} & \dots & 0 & x_{N-1\,N} & 0 & \dots \\ -x_{1\,N} & -x_{2\,N} & -x_{3\,N} & \dots & -x_{N-1\,N} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots \end{bmatrix}.$$

Namely, $x(N)_{i\,j} = x_{i\,j}$, for any $i \le n$, $j \le n$; $x(N)_{i\,j} = 0$, otherwise.

Let's introduce

$$S_i = \sum_{j=1}^{N} x(N)_{i\,j}. \tag{4.54}$$

$$x(N,k) = x(N) + {}^{0}x(N,k), \text{where } x(N,k) \in \widetilde{Mat}_{as}(HS), \tag{4.55}$$

$${}^{0}x(N,k) = \begin{bmatrix} 0 & \dots & 0 & | & \frac{-S_1}{k} & \frac{-S_1}{k} & \dots & \frac{-S_1}{k} & 0 & \dots \\ \vdots & \ddots & \vdots & | & \frac{-S_2}{k} & \frac{-S_2}{k} & \dots & \frac{-S_2}{k} & 0 & \dots \\ 0 & \dots & 0 & | & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ - & - & - & | & \frac{-S_N}{k} & \frac{-S_N}{k} & \dots & \frac{-S_N}{k} & 0 & \dots \\ \frac{S_1}{k} & \frac{S_2}{k} & \dots & \frac{S_N}{k} & 0 & 0 & \dots & 0 & 0 & \dots \\ \frac{S_1}{k} & \frac{S_2}{k} & \dots & \frac{S_N}{k} & 0 & 0 & \dots & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{S_1}{k} & \frac{S_2}{k} & \dots & \frac{S_N}{k} & 0 & 0 & \dots & 0 & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Namely,

$$
\begin{aligned}
&{}^0x(N,k)_{i\,j} = 0, \text{ if } i \le N \text{ and } j \le N;\\
&{}^0x(N,k)_{i\,j} = \frac{-S_i}{k} \text{ if } i \le N \text{ and } N < j \le N+k;\\
&{}^0x(N,k)_{i\,j} = \frac{S_i}{k} \text{ if } j \le N \text{ and } N < i \le N+k;\\
&x(N)_{i\,j} = 0, \text{ otherwise.}
\end{aligned}
\tag{4.56}
$$

It follows from 4.56 that

$$
\|x(N)-x(N,k)\|_{HS}^2 = \\
(\Sigma_{i=1}^k \tfrac{|S_1|^2}{k^2} + \Sigma_{i=1}^k \tfrac{|S_2|^2}{k^2} + \ldots + \Sigma_{i=1}^k \tfrac{|S_n|^2}{k^2}) + \tfrac{k}{k^2}(|S_1|^2 + \ldots + |S_n|^2) = \\
\tfrac{2}{k}|S_1|^2 + \ldots + |S_n|^2
$$

Note that there exists k such that

$$
\|x(N)-x(N,k)\|_{HS} < \epsilon/2. \tag{4.57}
$$

Thus, by 4.53 and 4.57, we obtain

$$
\|x-x(N,k)\| \le \|x-x(N)\| + \|x(N)-x(N,k)\| < \epsilon/2+\epsilon/2 = \epsilon
$$

□

Remark 53. $\overline{\widetilde{Mat}_{as}^{fin}(HS)} = \overline{\widetilde{Mat}_{as}(HS)} = Mat_{as}(HS)$.

Theorem 54. *The restriction of ρ to $Mat_{as}(HS)$ is an irreducible representation.*

Proof. Let's take an arbitrary nonzero $A = [A_{i\,j}] \in Mat_{as}(HS)$.
Since A is nonzero, then there exists $A_{i\,j} \ne 0$ and $s \in S_\infty$:

$$
(\rho(s)(A))_{1\,3} = (\rho((3\ j))(\rho((1\ i))(A)))_{1\,3} \ne 0. \tag{4.58}
$$

If $A_{1\,3} = 0$, then we replace A by $\rho(s)(A)$ and consider the matrix:

$$
{}^1A \;=\; A - \rho((1\ 2))(A) \tag{4.59}
$$

Consider cases:
case1.1 If $A_{1\,3} \ne A_{2\,3}$, then

$$
{}^2A = {}^1A. \tag{4.60}
$$

Otherwise:
case1.2 If $A_{1\,3} = A_{2\,3}$, then consider

$$
{}^2A = \rho((1\ l))(A) - \rho((1\ 2))(\rho((1\ l))(A)), \tag{4.61}
$$

where $l \in \mathbb{N}$ such that $a_{1\,3} \ne a_{l\,3}$. Since $A \ne 0$ and $A \in \widetilde{Mat}_{as}(HS)$, then there exists such $l > 3$ that $a_{1\,3} \ne a_{l\,3}$.

Let ${}^2A$ is a matrix obtained after previous steps, then consider cases:
case2.1 If $({}^2A)_{1\,3} \neq ({}^2A)_{1\,4}$, then consider:

$$ {}^3A = {}^2A - \rho((3\;4))({}^2A). \tag{4.62}$$

Otherwise:
case2.2 If $({}^2A)_{1\,3} = ({}^2A)_{1\,4}$, then consider:

$$ {}^3A = \rho((3\;k))({}^2A) - \rho((3\;4))(\rho((3\;k))({}^2A)). \tag{4.63}$$

Since ${}^2A \neq 0$ and ${}^2A \in \widetilde{Mat}_{as}(HS)$, then there exists such $k > 4$ that $({}^2A)_{1\,3} \neq ({}^2A)_{1\,k}$.

Thus, we obtain

$$ {}^3A = \lambda \begin{bmatrix} 0 & 0 & 1 & -1 & 0 & \dots \\ 0 & 0 & -1 & 1 & 0 & \dots \\ -1 & 1 & 0 & 0 & 0 & \dots \\ 1 & -1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. $$

Using Theorem 27 we obtain that $v \in Lin\{\rho(G)(A)\}$ for every element $v \in \widetilde{Mat}^{fin}_{as}(HS)$. Thus, $\widetilde{Mat}^{fin}_{as}(HS) \subset Lin\{\rho(G)(A)\}$. Using Remark 53 and ρ-invariance of $Mat_{as}(HS)$, we obtain that $Mat_{as}(HS) = \overline{\widetilde{Mat}_{as}(HS)} = \overline{\widetilde{Mat}^{fin}_{as}(HS)} = \overline{Lin\{\rho(G)(A)\}}$. Thus, the restriction of ρ to $Mat_{as}(HS)$ is an irreducible representation. $\square$

5 references

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