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Анотації

Сарапін Р.В. Розподіл власних значень загального деформованого ансамблю Жинібра.

Розглянемо матрицю $n \times n$ $X_n = A_n + H_n$, де A_n – матриця $n \times n$ (детерміністична або випадкова), а H_n – матриця $n \times n$, незалежна від A_n, взята з комплексного ансамблю Жинібра. В роботі досліджується граничний розподіл власних значень X_n. В [29] було показано, що розподіл власних значень X_n збігається до деякої детерміністичної міри. Ця міра була відома для випадку $A_n = 0$. За деяких загальних умов збіжності на A_n, в цій роботі доведено формулу для густини граничної міри. Також отримано оцінку на швидкість збіжності розподілу. Використаний підхід ґрунтується на суперсиметричному інтегруванні.

Sarapin R.V. Limiting eigenvalue distribution of the general deformed Ginibre ensemble.

Consider the $n \times n$ matrix $X_n = A_n + H_n$, where A_n is a $n \times n$ matrix (either deterministic or random) and H_n is a $n \times n$ matrix independent from A_n drawn from complex Ginibre ensemble. We study the limiting eigenvalue distribution of X_n. In [29] it was shown that the eigenvalue distribution of X_n converges to some deterministic measure. This measure was known for the case $A_n = 0$. Under some general convergence conditions on A_n we prove a formula for the density of the limiting measure. We also obtain an estimation on the rate of convergence of the distribution. The approach used here is based on supersymmetric integration.

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Introduction

General information

Let H_n be a $n \times n$ random matrix be i.i.d. complex random entries h_{ij} satisfying $\mathbb{E}\{h_{ij}\} = 0$ and $\mathbb{E}\{|h_{ij}|^2\} = 1/n$. Consider a random $n \times n$ matrix

$$X_n = A_n + H_n,$$

where A_n is either deterministic or random $n \times n$ matrix with entries independent of h_{ij}. The matrices X_n form the so called deformed Ginibre ensemble. Matrices of such form play a significant role in communication theory, where A_n and H_n are considered to be the signal matrix and the noise matrix respectively. For the purposes of that theory, the behaviour of the smallest singular value $\sigma_1(X_n) = \lambda_1(X_n X_n^*)$ of X_n has been extensively studied.

This paper is concerned with the limiting eigenvalue distribution of X_n. The problem of deriving the limiting distribution of eigenvalues for different random matrix ensembles is one of the fundamental problems in Random Matrix Theory. The first result on this way was obtained by Wigner [30] for $n \times n$ hermitian matrices with i.i.d. gaussian random entries. This result was later generalized for the case of arbitrary i.i.d. entries with mean 0 and variance $1/n$ (see [21]). The first conjecture on the eigenvalue distribution for non-hermitian matrices, known as *Circular Law Conjecture*, was posed in 1950's. The conjecture stated that for $n \times n$ matrices with i.i.d. random entries with mean 0 and variance $1/n$, the limiting eigenvalue distribution on the complex plane is given by the following density:

$$\rho(z) = \begin{cases} \frac{1}{\pi}, & |z| \leq 1, \\ 0, & |z| > 1, \end{cases}$$

i.e. the eigenvalues are distributed uniformly on the unit disk. The case of complex gaussian entries was obtained by Mehta [19] in 1967. However, it turned out to be much harder to generalize the result for an arbitrary distribution of the entries. In certain partial cases, Circular Law Conjecture was verified in [10], [14, 15], [1, 2], [16, 17], [20], [28], and was finally proven by Tao, Vu, Krishnapur in [29] in 2010.

Deformed Ginibre ensemble is in fact a generalisation of the non-hermitian ensemble mentioned above. In case of deterministic A_n, Tao, Vu, Krishnapur proved that the limiting eigenvalue distribution exists (see [29, Theorem 1.23]) and that it does not depend on the distribution of h_{ij} (see [29, Theorem 1.7]). The limiting distribution has been studied using free probability theory. Śniady [27] showed that if A_n converges in $*$-moments to some operator x_0, then the limiting measure of X_n is equal to the Brown measure of $x_0 + c$, where c is Voiculescu's circular operator which is $*$-free from x_0. This Brown measure was obtained in particular cases of self-adjoint or unitary c_0 (see [18]), normal c_0 (see [5]), and for the general case in the recent preprint [31].

In this paper we obtain the density of the limiting distribution under different conditions on A_n. We use another approach based on supersymmetric integration. Supersymmetry techniques have been successfully applied to different problems in Random Matrix Theory, see [3], [8], [9], [24], [25], [26], [11], [23], [6] for the details.

Basic notations and main results

Set $X_n = A_n + H_n$, where H_n is a random $n \times n$ matrix with i.i.d. complex gaussian entries $\{h_{ij}\}_{i,j=1}^n$ satisfying the following conditions:

$$\mathbb{E}\{h_{ij}\} = 0, \quad \mathbb{E}\{h_{ij}^2\} = 0, \quad \mathbb{E}\{|h_{ij}|^2\} = 1/n,$$

and A_n is $n \times n$ matrix which is either deterministic or random with entries $\{a_{ij}\}_{i,j=1}^n$ independent of h_{ij}. Denote

$$\begin{aligned} Y_0(z) &= (A_n - z)(A_n - z)^*, \quad Y(z) = (X_n - z)(X_n - z)^*, \\ \widetilde{Y}_0(z) &= (A_n - z)^*(A_n - z), \quad \widetilde{Y}(z) = (X_n - z)^*(X_n - z). \end{aligned} \tag{0.1}$$

Also we denote $\mathrm{tr}_n B = \dfrac{1}{n} \operatorname{Tr} B$ for any $n \times n$ matrix B. Our goal is to find the limit of *normalised counting measure* (NCM) of X_n. In order to be able to derive the result, we put the following constraints on A_n:

(C1) The NCM $\nu_{n,z}$ of $Y_0(z)$ converges weakly to some deterministic measure ν_z for almost all $z \in \mathbb{C}$.

(C2) There exists some $M > 0$ such that for some $d > 0$ and

$$\Omega_n^{(0)} = \{\omega \in \Omega \mid n^{-1} \sum_{i,j=1}^n |a_{ij}|^2 < M\}$$

we have $\mathrm{Prob}\{\Omega_n^{(0)}\} \geq 1 - n^{-1-d}$. Here and below Ω stands for the probability space with respect to A_n.

(C3) Denote $\sigma_0 = \{z \mid 0 \in \operatorname{supp} \nu_{n,z}\}$. Let σ_ϵ be the ϵ-neighbourhood of σ_0 and

$$\begin{aligned} \Omega_{\epsilon,\kappa,n}^{(1)} &= \{\omega \in \Omega \mid \inf_{z \notin \sigma_\epsilon} \operatorname{dist}(0, \operatorname{supp} \nu_{n,z}) \geq \kappa\}, \\ \Omega_{\epsilon,C,n}^{(2)} &= \{\omega \in \Omega \mid \sup_{z \notin \sigma_\epsilon} \Big| \mathrm{tr}_n Y_0^{-1}(z) - \int \lambda^{-1} \, d\nu_z(\lambda) \Big| < C n^{-d_0}\} \end{aligned}$$

for some $d_0 > 0$. Then for some $d > 0$ and for all $\epsilon > 0$ there exist $\kappa(\epsilon) > 0$, $C(\epsilon) > 0$ satisfying

$$\mathrm{Prob}\{\Omega^{(1)}_{\epsilon,\kappa(\epsilon),n} \cap \Omega^{(2)}_{\epsilon,C(\epsilon),n}\} > 1 - Cn^{-1-d}.$$

(C4) There exist $d_1 > 0$, $\varrho_0, \epsilon_0 > 0$ such that if

$$\Omega^{(3)}_n = \{\omega \in \Omega \mid \inf_{z\in\sigma_{\epsilon_0}} \mathrm{tr}_n(Y_0(z) + \varrho_0^2)^{-1} > 1 + d_1\}$$

then $\mathrm{Prob}\{\Omega^{(3)}_n\} > 1 - Cn^{-1-d}$ for some $C > 0, d > 0$.

(C5) There exist functions $T_1(z, x)$ and $T_2(z, x)$ such that
$\lim_{n\to\infty} \mathrm{tr}_n(A_n - z)(Y_0(z) + x)^{-2} = T_1(z, x)$ and
$\lim_{n\to\infty} \mathrm{tr}_n(Y_0(z) + x)^{-1}(\widetilde{Y}_0(z) + x)^{-1} = T_2(z, x)$ almost surely for all $x > 0$, $z \in \mathbb{C}$.

Remark 0.1. *In case when A_n is deterministic, in (C2), (C3), (C4) we assume that the inequalities in the definitions of $\Omega^{(0)}_n, \Omega^{(1)}_{\epsilon,\kappa,n}, \Omega^{(2)}_{\epsilon,C,n}, \Omega^{(3)}_n$ hold for large enough n.*

Remark 0.2. *Notice that Borel-Cantelli lemma together with (C1)–(C5) implies that*

$$\mathrm{Prob}\{\exists n_0\colon \omega \in \Omega^{(0)}_n \cup \Omega^{(1)}_{\epsilon,\kappa,n} \cup \Omega^{(2)}_{\epsilon,C,n} \cup \Omega^{(3)}_n \ \forall n \geq n_0\} = 1.$$

This allows us to consider only the case $\omega \in \Omega^{(0)}_n \cup \Omega^{(1)}_{\epsilon,\kappa,n} \cup \Omega^{(2)}_{\epsilon,C,n} \cup \Omega^{(3)}_n$.

Denote μ_{n,X_n} the NCM of X_n. In case if A_n is deterministic, according to [29, Theorem 1.23], there exists some deterministic measure μ such that μ_{n,X_n} converges to μ weakly. If A_n is random, we may fix $\omega \in \Omega$ (i.e. fix some sequence of A_n as $n \in \mathbb{N}$) and conclude that there exists some deterministic (with respect to H_n) measure μ_ω such that μ_{n,H_n} converges to μ_ω weakly.

Consider the set

$$D = \sigma_0 \cup \{z \in \mathbb{C} \setminus \sigma_0 \colon \int \lambda^{-1}\, d\nu_z(\lambda) \geq 1\}. \tag{0.2}$$

According to [4], the additional assumption $\operatorname{supp} \mu_\omega = \{z \mid 0 \in \operatorname{supp} \eta_z\}$, where η_z is the limit of NCM of $Y(z) = (X_n - z)(X_n - z)^*$, guarantees that $\operatorname{supp} \mu_\omega = D$. However, we will not require the assumption above to conclude that $\operatorname{supp} \mu_\omega = D$.

Observe that (C4) implies $\sigma_{\epsilon_0} \subset D$. Moreover, due to (C3), $\int \lambda^{-1}\, d\nu_z(\lambda) = \lim_{n\to\infty} \operatorname{tr}_n Y_0(z)^{-1}$ for $z \in \mathbb{C} \setminus \sigma_{\epsilon_0}$ is a smooth function in z, which means that

$$\partial D = \{z \in \mathbb{C} \colon \int \lambda^{-1}\, d\nu_z(\lambda) = 1\} \tag{0.3}$$

and consists of several curves enclosing σ_0.

We are ready to formulate the main result of the paper.

Theorem 0.1. *Suppose that $X_n = A_n + H_n$ defined as above satisfies (C1)–(C5). Set*

$$\rho(z) = \begin{cases} \dfrac{1}{\pi}\Big(\dfrac{|T_1(z, x_0^2)|^2}{\int (\lambda + x_0^2)^{-2}\, d\nu_z(\lambda)} + x_0^2 \cdot T_2(z, x_0^2)\Big), & z \in \operatorname{Int} D; \\ 0\,, & z \in \mathbb{C} \setminus D, \end{cases} \tag{0.4}$$

with $x_0 = x_0(z) > 0$ satisfying the equation $\int (\lambda + x_0^2)^{-1}\, d\nu_z(\lambda) = 1$. Then, for any $\omega \in \Omega$ and for any smooth compactly supported function $h(z)$ we have

$$\int h(z)\, d\mu_\omega(z) = \int h(z) \rho(z)\, dz.$$

The above result yields that μ_ω does not actually depend on ω. In other words, there is a deterministic measure μ given by its density $\rho(z)$ defined in (0.4) which is a limit of NCM of X_n. More precisely,

Corollary 0.2. *Suppose that $X_n = A_n + H_n$ defined as above satisfies (C1)–(C5). Then the normalised counting measure of X_n converges weakly to the measure μ with density $\rho(z)$ defined in (0.4).*

We also obtain an estimation on the rate of weak convergence of NCM. Naturally, we need some additional assumptions on the rate of the convergence of $\nu_{n,z}$:

(C6) For $\kappa > 0$, $C > 0$ denote

$$\Omega^{(4)}_{\kappa,C,n} = \{\omega \in \Omega \mid \sup_{z\in D,\, x\geq\kappa} \Big|\mathrm{tr}_n(Y_0(z)+x)^{-1} - \int \frac{d\nu_z(\lambda)}{\lambda + x}\Big| < Cn^{-1/5}\};$$
$$\Omega^{(5)}_{\kappa,C,n} = \{\omega \in \Omega \mid \sup_{z\in D,,x\geq\kappa} \Big|\mathrm{tr}_n(Y_0(z)+x)^{-2} - \int \frac{d\nu_z(\lambda)}{(\lambda + x)^2}\Big| < Cn^{-1/5}\};$$
$$\Omega^{(6)}_{\kappa,C,n} = \{\omega \in \Omega \mid \sup_{z\in D,\, x\geq\kappa} \big|\mathrm{tr}_n(A_n - z)(Y_0(z)+x)^{-2} - - T_1(z,x)\big| < Cn^{-1/5}\};$$
$$\Omega^{(7)}_{\kappa,C,n} = \{\omega \in \Omega \mid \sup_{z\in D,\, x\geq\kappa} \big|\lim_{n\to\infty} \mathrm{tr}_n(Y_0(z)+x)^{-1}(\widetilde{Y}_0(z)+x)^{-1} - - T_2(z,x)\big| < Cn^{-1/5}\};$$

Then for some $d > 0$ and for all $\kappa > 0$ there exists $C(\kappa) > 0$ satisfying

$$\mathrm{Prob}\Big\{\bigcap_{j=4}^{7} \Omega^{(j)}_{\kappa,C(\kappa),n}\Big\} > 1 - Cn^{-1-d}.$$

We get the following result:

Theorem 0.3. *Suppose that $X_n = A_n + H_n$ defined as above satisfies (C1)–(C6). Let $z_1, \ldots, z_n$ be the eigenvalues of X_n and $h(z)$ be a smooth function on $\mathbb{C}$ with a compact support E such that $\mathrm{dist}(E, \partial D) \geq d > 0$. Then*

$$\Big|\mathbb{E}\Big\{\frac{1}{n}\sum_{j=1}^{n} h(z_j)\Big\} - \int h(z)\rho(z)\,dz\Big| \leq Cn^{-1/5}.$$

Structure of the paper

The paper is organised as follows. In Chapter 1 we introduce

$$\rho_{\varepsilon,n,\omega}(z) = \frac{1}{4\pi n}\Delta\,\mathbb{E}_{H_n}\Big\{\log\det(Y(z)+\varepsilon^2)\Big\}$$

and show that it is sufficient to compute $\rho_{\varepsilon,\omega}(z) = \lim\limits_{n\to\infty}\rho_{\varepsilon,n,\omega}(z)$ and $\rho_\omega(z) = \lim\limits_{\varepsilon\to 0}\rho_{\varepsilon,\omega}(z)$ to obtain the density $\rho_\omega(z)$ of μ_ω. In Chapter 2 we derive the integral representation of $\rho_{\varepsilon,n,\omega}(z)$. In Chapter 3 we perform asymptotic analysis of $\rho_{\varepsilon,n,\omega}(z)$ using its integral representation. In Chapter 4 we get the final expression for $\rho_\omega(z)$. Chapter 5 consists of the proof of some technical proposition from Section 2. Finally, in Chapter 6 we derive the rate of convergency for NCM of X_n.

Chapter 1
Strategy for computation of the limiting density

Let $z_1, \dots, z_n$ be the eigenvalues of X_n. According to the standard potential theory,

$$\rho_{n,X_n}(z) = \frac{1}{4\pi n}\Delta \log\det Y(z) = \frac{1}{4\pi n}\Delta \sum_{j=1}^{n} \log|z - z_j|^2$$

is the density of NCM μ_{n,X_n} of X_n, i.e.

$$\int h(z)\cdot \frac{1}{4\pi n}\Delta \log\det Y(z)\, \mathbf{d}^2 z = \frac{1}{n}\sum_{j=1}^{n} h(z_j) = \int h(z)\, d\mu_{n,X_n} \tag{1.1}$$

for an arbitrary smooth function $h(z)$ with compact support, where $Y(z)$ is defined in (0.1), Δ is a two-dimensional Laplacian on $\mathbb{C}$ and $\mathbf{d}^2 z$ is a standard two-dimensional Lebesgue measure on $\mathbb{C}$. The idea of this paper is to compute some 'regularisation' of this density and then take the limit as $n \to \infty$.

One can consider $\widetilde{\rho}_{\varepsilon,n,X_n}(z) = \dfrac{1}{4\pi n}\displaystyle\sum_{j=1}^{n} \Delta \log(|z - z_j|^2 + \varepsilon^2)$ and show that

$$\lim_{\varepsilon\to 0}\int h(z)\widetilde{\rho}_{\varepsilon,n,X_n}(z)\, \mathbf{d}^2 z = \int h(z)\rho_{n,X_n}(z)\, \mathbf{d}^2 z$$

uniformly in n, which allows to find the density of the limiting measure μ_ω as the double limit $\lim\limits_{\substack{\varepsilon\to 0\\ n\to\infty}} \widetilde{\rho}_{\varepsilon,n,X_n}$. This approach is not suitable for our problem since $\displaystyle\sum_{j=1}^{n} \log(|z - z_j|^2 + \varepsilon^2) \neq \log\det(Y(z) + \varepsilon^2)$ in general and there is no simple way to express $\widetilde{\rho}_{\varepsilon,n,X_n}$ in terms of X_n. Instead, we introduce the following regularisation of $\mathbb{E}_{H_n}\{\rho_{n,X_n}(z)\}$:

$$\rho_{\varepsilon,n,\omega}(z) = \frac{1}{4\pi n}\Delta\,\mathbb{E}_{H_n}\Big\{\log\det(Y(z)+\varepsilon^2)\Big\}. \tag{1.2}$$

It is harder to obtain the uniform convergence of $\displaystyle\int h(z)\rho_{\varepsilon,n,\omega}(z)\,\mathbf{d}^2z$. However, the following result holds:

Proposition 1.1. *Suppose that $X_n = A_n + H_n$ defined as above satisfies (C1)–(C5). Let $z_1,\dots,z_n$ be the eigenvalues of X_n and $h(z)$ be a smooth compactly supported function on $\mathbb{C}$. Then there exists $n_0 \in \mathbb{N}$ such that $\displaystyle\lim_{\varepsilon\to 0}\int h(z)\rho_{\varepsilon,n,\omega}(z)\,\mathbf{d}^2z = \mathbb{E}_{H_n}\Big\{\frac{1}{n}\sum_{j=1}^{n} h(z_j)\Big\}$ uniformly in n for $n \geq n_0$.*

This proposition is proven in Chapter 5 using the integral representation of $\partial_\varepsilon\,\mathbb{E}_{H_n}\{\log\det(Y(z)+\varepsilon^2)\}$. It allows to change the order of the limits while computing the density of μ_ω. More precisely,

Proposition 1.2. *Let $h(z)$ be a smooth function on $\mathbb{C}$ with a compact support E. Suppose that for almost all $z \in E$ there exists $\displaystyle\lim_{n\to\infty}\rho_{\varepsilon,n,\omega}(z) = \rho_{\varepsilon,\omega}(z)$ such that $|\rho_{\varepsilon,\omega}(z)| \leq C$ for $0 < \varepsilon \leq \varepsilon_0$ and $z \in E$, and there exists $\displaystyle\lim_{\varepsilon\to 0}\rho_{\varepsilon,\omega}(z) = \rho_\omega(z)$ for $z \in E$. Then*

$$\int h(z)\rho_\omega(z)\,\mathbf{d}^2z = \int h(z)d\mu_\omega(z).$$

Proof.

Again, let $\{z_1, z_2, \dots, z_n\}$ be the set of eigenvalues of X_n. Recall that $\mu_{n,\omega} \to \mu_\omega$ weakly, which means that $\displaystyle\int h(z)\,d\mu_{n,\omega} = \frac{1}{n}\sum_{j=1}^{n} h(z_j) \to \int h(z)\,d\mu_\omega$ almost surely. Moreover, $\displaystyle\Big|\frac{1}{n}\sum_{j=1}^{n} h(z_j)\Big| \leq \max_{z\in E}|h(z)| < \infty$. Then dominated convergence theorem gives us

$$\lim_{n\to\infty} \mathbb{E}_{H_n}\Big\{\frac{1}{n}\sum_{j=1}^{n} h(z_j)\Big\} = \int h(z)\, d\mu_\omega.$$

One can check: $\rho_{\varepsilon,n,\omega}(z) = \dfrac{1}{\pi}\varepsilon^2\, \mathbb{E}_{H_n}\Big\{\mathrm{tr}_n(Y(z)+\varepsilon^2)^{-1}(\widetilde{Y}(z)+\varepsilon^2)^{-1}\Big\} \le \dfrac{1}{\pi\varepsilon^2}$. This bound together with the assumptions of the theorem give us

$$\lim_{n\to\infty} \int h(z)\rho_{\varepsilon,n,\omega}(z)\, \mathbf{d}^2 z = \int h(z)\rho_{\varepsilon,\omega}(z)\, \mathbf{d}^2 z,$$
$$\lim_{\varepsilon\to 0} \int h(z)\rho_{\varepsilon,\omega}(z)\, \mathbf{d}^2 z = \int h(z)\rho_{\omega}(z)\, \mathbf{d}^2 z.$$

Now recall the following fact about interchanging limits of functions known as Moore-Osgood theorem. Let $f(n,\varepsilon)$, $g_1(n)$, $g_2(\varepsilon)$ be functions such that $\lim\limits_{\varepsilon\to+0} f(n,\varepsilon) = g_1(n)$ uniformly in n for $n \ge n_0$ and $\lim\limits_{n\to\infty} f(n,\varepsilon) = g_2(\varepsilon)$ for all $\varepsilon \in (0,\varepsilon_0)$. Then there exist limits $\lim\limits_{n\to\infty} g_1(n)$ and $\lim\limits_{\varepsilon\to+0} g_2(\varepsilon)$ and they are equal, i.e.

$$\lim_{n\to\infty}\lim_{\varepsilon\to+0} f(n,\varepsilon) = \lim_{\varepsilon\to+0}\lim_{n\to\infty} f(n,\varepsilon).$$

Apply this result to $f(n,\varepsilon) = \displaystyle\int h(z)\rho_{\varepsilon,n,\omega}(z)\, \mathbf{d}^2 z$, $g_1(n) = \mathbb{E}_{H_n}\Big\{\dfrac{1}{n}\displaystyle\sum_{j=1}^{n} h(z_j)\Big\}$, $g_2(\varepsilon) = \displaystyle\int h(z)\rho_{\varepsilon,\omega}(z)\, \mathbf{d}^2 z$. Proposition 1.1 together with the convergences above yield that $\displaystyle\int h(z)\rho_{\omega}(z)\, \mathbf{d}^2 z = \int h(z)\, d\mu_\omega$.

□

Remark 1.1. *Comparing the proposition above with Theorem 0.1, we see that it is sufficient to prove the following facts:*

- *there exist limits* $\lim\limits_{n\to\infty} \rho_{\varepsilon,n,\omega}(z) = \rho_{\varepsilon,\omega}(z)$ *and* $\lim\limits_{\varepsilon\to 0} \rho_{\varepsilon,\omega}(z) = \rho_\omega(z)$ *for almost all* $z \in \mathbb{C}$*;*
- $\rho_{\varepsilon,\omega}(z)$ *is bounded uniformly in* $\varepsilon \le \varepsilon_0$ *and almost all* $z\colon |z| \le C$*;*

- *$\rho_\omega(z) = \rho(z)$ where $\rho(z)$ is given by (0.4);*

- *Proposition 1.1 holds.*

Below the formula for $\rho_{\varepsilon,n,\omega}$ is rewritten to be more suitable for supersymmetric integration. We use the trick introduced by Fyodorov, Sommers in [12]. Recall that $\Delta = 4\partial_{\bar z}\partial_z$, where ∂_z and $\partial_{\bar z}$ are Wirtinger derivatives defined as $\partial_z f(z) = \dfrac{1}{2}(\partial_x - i\partial_y)f(x+iy)$, $\partial_{\bar z} f(z) = \dfrac{1}{2}(\partial_x + i\partial_y)f(x+iy)$. Straightforward computation shows that

$$\partial_{\bar z}\partial_z \log\det(Y(z)+\varepsilon^2) = \partial_{\bar z}\left(\partial_{z_1}\frac{\det(Y(z_1)+\varepsilon^2)}{\det(Y(z)+\varepsilon^2)}\right)\Big|_{z_1=z}. \tag{1.3}$$

This identity together with (1.2) implies

$$\rho_{\varepsilon,n,\omega}(z) = \frac{1}{\pi n}\partial_{\bar z}\Big(\big(\partial_{z_1}\mathcal{Z}(\varepsilon,\varepsilon,z,z_1)\big)\big|_{z_1=z}\Big), \tag{1.4}$$

where

$$\mathcal{Z}(\varepsilon,\varepsilon_1,z,z_1) = \mathbb{E}_{H_n}\left\{\frac{\det(Y(z_1)+\varepsilon_1^2)}{\det(Y(z)+\varepsilon^2)}\right\}. \tag{1.5}$$

Remark 1.2. *In order to find $\rho_{\varepsilon,n,\omega}(z)$, we need to compute $\mathcal{Z}(\varepsilon,\varepsilon_1,z,z_1)$ only for $\varepsilon_1 = \varepsilon$. However, the general formula for arbitrary $\varepsilon,\varepsilon_1 > 0$ will be used to prove Proposition 1.1.*

Chapter 2
Integral representations of $\mathcal{Z}(\varepsilon, \varepsilon_1, z, z_1)$ and $\rho_{\varepsilon,n,\omega}(z)$

2.1. Representation of $\mathcal{Z}(\varepsilon, \varepsilon_1, z, z_1)$

The next theorem gives us an integral representation of $\mathcal{Z}(\varepsilon, \varepsilon_1, z, z_1)$ defined in (1.5).

Proposition 2.1. *We have*

$$\begin{aligned}\mathcal{Z}(\varepsilon, \varepsilon_1, z, z_1) = &\frac{2n^3}{\pi^3}\int_0^\infty dR \int_{-\infty}^{\infty} dv\, du_1\, du_2\, ds \int_L dt \cdot \frac{R}{\sqrt{v^2+4R}} \times \\ &\times \varphi_n(u_1^2+u_2^2, s^2-t^2, z, z_1)\times \\ &\times \exp\Big\{n\Big(\mathcal{L}_n(z_1, u_1^2+u_2^2) - (u_1+\varepsilon_1)^2 - u_2^2\Big)\Big\}\times \\ &\times \exp\Big\{-n\Big(\mathcal{L}_n(z, s^2-t^2) + (t-i\varepsilon)^2 + (R+it+\varepsilon)^2 + \varepsilon v^2\Big)\Big\},\end{aligned} \tag{2.1}$$

with $L := \mathbb{R} + \varepsilon_0 i, \quad \varphi_n(x,y,z,z_1) = \varphi_{n,1}(x,y,z,z_1) - \dfrac{1}{n}\varphi_{n,2}(x,y,z,z_1),$

$$\begin{aligned}\varphi_{n,1}(x,y,z,z_1) = &(1-\mathrm{tr}_n(A_n-z_1)^*G(z_1,x)G(z,y)(A_n-z))\times \\ &\times(1-\mathrm{tr}_n G(z_1,x)(A_n-z_1)(A_n-z)^*G(z,y)) - \\ &- xy\,(\mathrm{tr}_n G(z_1,x)G(z,y))\Big(\mathrm{tr}_n \widetilde{G}(z_1,x)\widetilde{G}(z,y)\Big);\\ \varphi_{n,2}(x,y,z,z_1) = &\\ = y\cdot &\mathrm{tr}_n[G(z_1,x)(A_n-z_1)\widetilde{G}(z,y)(A_n-z_1)^*G(z_1,x)G(z,y)] + \\ + x\cdot &\mathrm{tr}_n[G(z_1,x)G(z,y)(A_n-z)\widetilde{G}(z_1,x)(A_n-z)^*G(z,y)],\end{aligned} \tag{2.2}$$

where

$$\begin{aligned}
\mathcal{L}_n(z,x) &= \frac{1}{n}\log\det\big(Y_0(z)+x\big) = \frac{1}{n}\log\det\big((z-A_n)(\bar z - A_n^*)+x\big),\\
G(z,x) &= (Y_0(z)+x)^{-1} = ((z-A_n)(\bar z - A_n^*)+x)^{-1},\\
\widetilde{G}(z,x) &= (\widetilde{Y}_0(z)+x)^{-1} = ((\bar z - A_n^*)(z-A_n)+x)^{-1}.
\end{aligned} \tag{2.3}$$

Proof. One can prove this similarly to [23, Proposition 2.1]. In fact, the integral representation of $\mathcal{Z}(\varepsilon,\varepsilon_1,z,z)$ (which is $\mathcal{Z}(\varepsilon,\varepsilon_1)$ in [23]) is obtained there, and then the identity is differentiated with respect to ε_1 in order to get $T(z,\varepsilon)$. One can obtain the representation of $\mathcal{Z}(\varepsilon,\varepsilon_1,z,z_1)$ using the same strategy.

At the final step we make the following change of variables:

$$(t_1,t_2)\to(s,t),\quad s=\frac{t_1-t_2}{2},\quad t=\frac{t_1+t_2}{2},\quad s\in\mathbb{R},\ t\in L,$$
$$(r_1,r_2)\to(v,R),\quad v=r_1-r_2,\quad R=r_1r_2,\quad v\in\mathbb{R},\ R\in[0,+\infty).$$

The Jacobians of these changes are $J_1=2$ and $J_2=\dfrac{1}{(v^2+4R)^{1/2}}$. It is easy to see that we obtain (2.1).

$\square$

Remark 2.1. *It follows from the proof of [23, Proposition 2.1] that for $z_1=z$ we have*

$$\begin{aligned}
\varphi_{n,1}(x,y,z,z) =&\Big(1-\mathrm{tr}_nG(z,y)+x\mathrm{tr}_nG(z,x)G(z,y)\Big)^2-\\
&-xy(\mathrm{tr}_nG(z,x)G(z,y))^2;\\
\varphi_{n,2}(x,y,z,z) =&(x-y)(\mathrm{tr}_nG(z,x)G^2(z,y)-x\mathrm{tr}_nG^2(z,x)G^2(z,y)).
\end{aligned}$$

2.2. Representation of $\rho_{\varepsilon,n,\omega}(z)$

Now we obtain an integral representation of $\rho_{\varepsilon,n,\omega}$ using (1.4) and the representation of $\mathcal{Z}(\varepsilon,\varepsilon_1,z,z_1)$. Introduce the functional

$$\begin{aligned}\langle f(u_1,u_2,t,s,z,z_1)\rangle = \frac{2n^3}{\pi^3}\int_0^\infty dR\int_{-\infty}^\infty dv\,du_1\,du_2\,ds\int_L dt\times\\ \times\frac{R}{\sqrt{v^2+4R}}\cdot f(u_1,u_2,t,s,z,z_1)\times\\ \times\exp\{-n((u_1+\varepsilon)^2+u_2^2+(t-i\varepsilon)^2+(R+it+\varepsilon)^2+\varepsilon v^2)\},\end{aligned}\tag{2.4}$$

We can use it to rewrite (2.1) as

$$\mathcal{Z}(\varepsilon,\varepsilon,z,z_1)=\Big\langle\,\varphi_n(z,z_1)e^{n\mathcal{F}_n(z,z_1)}\,\Big\rangle,\tag{2.5}$$

where $\mathcal{F}_n(z,z_1)=\mathcal{L}_n(z_1,u_1^2+u_2^2)-\mathcal{L}_n(z,s^2-t^2)$ and $\varphi_n(z,z_1)$ is a short notation for the function $\varphi_n(u_1^2+u_2^2,s^2-t^2,z,z_1)$ defined in (2.2). Further we will use the following trivial observation:

$$\Big\langle\,\varphi_n(z,z)e^{n\mathcal{F}_n(z,z)}\,\Big\rangle=\mathcal{Z}(\varepsilon,\varepsilon,z,z)=1.\tag{2.6}$$

Observe that $\partial_{\bar z}\big((\partial_{z_1}\mathcal{Z}(\varepsilon,\varepsilon,z,z_1))\big|_{z_1=z}\big)=\big((\partial_{\bar z}+\partial_{\bar z_1})\partial_{z_1}\mathcal{Z}(\varepsilon,\varepsilon,z,z_1)\big)\big|_{z_1=z}$. The last identity combined with (1.4) and (2.5) gives us the next result:

Proposition 2.2. *Consider the following integrals:*

$$\begin{aligned}\mathbf{I}_1&:=\Big\langle\,\partial_{z_1}\mathcal{F}_n(z,z_1)\cdot(\partial_{\bar z}+\partial_{\bar z_1})\big(\varphi_n(z,z_1)\,e^{n\mathcal{F}_n(z,z_1)}\big)\,\Big\rangle\Big|_{z_1=z};\\ \mathbf{I}_2&:=\Big\langle\,(\partial_{\bar z}+\partial_{\bar z_1})\partial_{z_1}\mathcal{F}_n(z,z_1)\cdot\varphi_n(z,z_1)\,e^{n\mathcal{F}_n(z,z_1)}\,\Big\rangle\Big|_{z_1=z};\\ \mathbf{I}_3&:=\Big\langle\,\partial_{z_1}\varphi_n(z,z_1)\cdot(\partial_{\bar z}+\partial_{\bar z_1})\mathcal{F}_n(z,z_1)\cdot\,e^{n\mathcal{F}_n(z,z_1)}\,\Big\rangle\Big|_{z_1=z};\\ \mathbf{I}_4&:=\frac{1}{n}\Big\langle\,(\partial_{\bar z}+\partial_{\bar z_1})\partial_{z_1}\varphi_n(z,z_1)\cdot e^{n\mathcal{F}_n(z,z_1)}\,\Big\rangle\Big|_{z_1=z}.\end{aligned}\tag{2.7}$$

Then $\rho_{\varepsilon,n,\omega}(z)=\dfrac{1}{\pi}(\mathbf{I}_1+\mathbf{I}_2+\mathbf{I}_3+\mathbf{I}_4)$.

Chapter 3

Asymptotic behaviour of $\rho_{\varepsilon,n,\omega}(z)$

In this chapter we perform an asymptotic analysis of $\mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3, \mathbf{I}_4$ as $n \to \infty$ for some fixed $\varepsilon > 0$ and $z \in \mathbb{C} \setminus \partial D$, which will give us the asymptotic behaviour of $\rho_{\varepsilon,n,\omega}(z)$ as $n \to \infty$.

Below we will write $\mathcal{L}_n(x)$, $G(x)$ and $\widetilde{G}(x)$ instead of $\mathcal{L}_n(z,x)$, $G(z,x)$ and $\widetilde{G}(z,x)$ to simplify the notations.

3.1. Preparations for the saddle point method

We will use the saddle point method to analyse certain integrals. To this end we study the solutions of some equations which will appear further as saddle points. Suppose that conditions (C1)–(C5) hold, and consider the following equations:

$$1 - \mathrm{tr}_n G(x^2) = \frac{\varepsilon}{x}, \tag{3.1}$$

$$1 - \int (\lambda + x^2)^{-1}\, d\nu_z(\lambda) = \frac{\varepsilon}{x}, \tag{3.2}$$

$$\mathrm{tr}_n G(x^2) = 1, \tag{3.3}$$

$$\int (\lambda + x^2)^{-1}\, d\nu_z(\lambda) = 1, \tag{3.4}$$

where $\varepsilon > 0$.

Let us fix a compact set E_{in} satisfying $E_{in} \subset \mathrm{Int}\, D$. First we study the solutions of the equations above while z takes values from E_{in}.

Proposition 3.1. *The equations* (3.2) *and* (3.4) *have exactly one positive root each for* $z \in \operatorname{Int} D$. *Moreover, there exists* $n_0 = n_0(E_{in})$ *such that for* $n \geq n_0$ *and* $z \in E_{in}$ *each of the equations* (3.1) *and* (3.3) *has exactly one positive root.*

Further we denote $x_{\varepsilon,n}$, x_ε, $x_{0,n}$ and x_0 the positive solutions of (3.1), (3.2), (3.3) and (3.4) correspondingly.

Proposition 3.2. *For a given* E_{in}, *there exist* $\kappa_0 = \kappa_0(E_{in}) > 0$, $n_1 = n_1(E_{in})$ *and* $\varepsilon_0 = \varepsilon_0(E_{in})$ *such that for all* $z \in E_{in}$, $n \geq n_1$ *and* $\varepsilon \leq \varepsilon_0$ *the following inequalities hold:*

1. $\kappa_0 \leq x_{0,n} \leq 1$, *2.* $x_{0,n} \leq x_{\varepsilon,n} \leq x_{0,n} + \dfrac{\varepsilon}{\kappa_0^2}$.

Also we have $\lim\limits_{\varepsilon\to 0} x_{\varepsilon,n} = x_{0,n}$, $\lim\limits_{n\to\infty} x_{0,n} = x_0$, $\lim\limits_{n\to\infty} x_{\varepsilon,n} = x_\varepsilon$, $\lim\limits_{\varepsilon\to 0} x_\varepsilon = x_0$. *Moreover, if (C6) holds, we have* $|x_{\varepsilon,n} - x_\varepsilon| \leq Cn^{-1/5}$ *for some* $C = C(E_{in})$.

Now let us fix a compact set E_{out} satisfying $E_{out} \subset \mathbb{C} \setminus \overline{D}$. We study the solutions of the equations above while z takes on values from E_{out}.

Proposition 3.3. *The equations* (3.1) *and* (3.2) *have exactly one positive root each, while* (3.4) *has no nonnegative roots for* $z \in \mathbb{C} \setminus \overline{D}$. *Moreover, there exists* $n_0 = n_0(E_{out})$ *such that for* $n \geq n_0$ *and* $z \in E_{out}$ *the equation* (3.3) *has no nonnegative roots.*

As before, we denote $x_{\varepsilon,n}$, x_ε the positive solutions of (3.1), (3.2) correspondingly.

Proposition 3.4. *For a given* E_{out}, *there exist* $K_0 = K_0(E_{out}) > 0, \kappa_0 = \kappa_0(E_{out}) > 0$ *and* $n_1 = n_1(E_{out})$ *such that for all* $z \in E_{out}$ *and* $n \geq n_1$ *the following inequality holds:* $\varepsilon(1+\kappa_0) \leq x_{\varepsilon,n} \leq \varepsilon(1+K_0)$. *Also, we have* $\lim\limits_{n\to\infty} x_{\varepsilon,n} = x_\varepsilon$.

One can easily show that Propositions 3.1–3.4 follow from the conditions (C1)–(C5) and Rouché's theorem.

Now consider a set $E_{\partial D} = \{z \in \mathbb{C} \mid \operatorname{dist}(z, \partial D) \le d\}$ for some small d. We have the following result:

Proposition 3.5. *For any $z \in E_{\partial D}$ there exists exactly one positive solution $x_{\varepsilon,n}$ of (3.1) and exactly one positive solution x_ε of (3.2). Moreover, one can find $C = C(E_{\partial D}) > 0$ and $c = c(E_{\partial D}) > 0$ such that*

$$\begin{aligned}
&c\varepsilon^{1/3} \le x_{\varepsilon,n} \le C, \quad \text{when } \operatorname{tr}_n G(0) \ge 1;\\
&(1+c)\varepsilon \le x_{\varepsilon,n} \le C\varepsilon^{1/3}, \quad \text{when } \operatorname{tr}_n G(0) < 1;\\
&c\varepsilon^{1/3} \le x_\varepsilon \le C, \quad \text{when } z \in D \cap E_{\partial D};\\
&(1+c)\varepsilon \le x_\varepsilon \le C\varepsilon^{1/3}, \quad \text{when } z \in E_{\partial D} \setminus D.
\end{aligned}$$

Proof. Suppose that $\operatorname{tr}_n G(0) \ge 1$. The upper bound $x_{\varepsilon,n}$ is obvious. Next, we have

$$\frac{\varepsilon}{x_{\varepsilon,n}} = 1 - \operatorname{tr}_n G(x_{\varepsilon,n}^2) \le \operatorname{tr}_n G(0) - \operatorname{tr}_n G(x_{\varepsilon,n}^2) = \operatorname{tr}_n G^2(\xi) \cdot x_{\varepsilon,n}^2 \le C x_{\varepsilon,n}^2$$

for some $\xi \in (0, x_{\varepsilon,n}^2)$, which gives us the lower bound.

Now suppose that $\operatorname{tr}_n G(0) < 1$. For the upper bound, observe that

$$\frac{\varepsilon}{x_{\varepsilon,n}} = 1 - \operatorname{tr}_n G(x_{\varepsilon,n}^2) > \operatorname{tr}_n G(0) - \operatorname{tr}_n G(x_{\varepsilon,n}^2) = \operatorname{tr}_n G^2(\xi) \cdot x_{\varepsilon,n}^2 > C x_{\varepsilon,n}^2.$$

For the lower bound, we have $\dfrac{\varepsilon}{x_{\varepsilon,n}} = 1 - \operatorname{tr}_n G(x_{\varepsilon,n}^2) < 1 - \operatorname{tr}_n G(C^2) < 1 - \kappa$ for some $\kappa > 0$, since $x_{\varepsilon,n} < C\varepsilon^{1/3} < C$. □

3.2. Integrals of the form $\left\langle g \cdot e^{n\mathcal{F}_n(z,z)} \right\rangle$, case $z \in \operatorname{Int} D$

In order to derive the asymptotics of $\mathbf{I}_k$, at first the integrals of more general form are studied. Set

$$\mathbf{I}_{g_n} = \left\langle g_n(u_1^2 + u_2^2, s^2 - t^2) e^{n\mathcal{F}_n(z,z)} \right\rangle,$$

where a sequence of complex-valued functions $g_n(x, y)$ satisfies the following conditions:

1. $g_n(x, y)$ are analytic in some neighbourhood of $(x_{\varepsilon,n}^2, x_{\varepsilon,n}^2)$;
2. $g_n(x, y)$ are bounded uniformly in n in a neighbourhood of $(x_{\varepsilon,n}^2, x_{\varepsilon,n}^2)$;
3. $\left\langle |g_n(u_1^2 + u_2^2, s^2 - t^2) e^{N_1\mathcal{F}_n(z,z)}| \right\rangle \leq C$ for some fixed N_1 and $C > 0$.

Our goal is to prove that

$$\mathbf{I}_{g_n} = \left.\frac{g_n(u_1^2 + u_2^2, s^2 - t^2)}{\varphi(u_1^2 + u_2^2, s^2 - t^2, z, z)}\right|_{\substack{u_1=-x_{\varepsilon,n},\ u_2=0,\\ t=ix_{\varepsilon,n},\ s=0}} + O(n^{-\alpha}), \tag{3.5}$$

where $x_{\varepsilon,n}$ is the positive root of (3.1) and φ_n is defined in (2.2).

In this section the case $z \in \operatorname{Int} D$ is studied. Let us fix some $z \in \operatorname{Int} D$ and take $E_{in} := \{z\}$. Then we can take $\kappa_0 = \kappa_0(E_{in}) > 0$ and use the result of Section 3.1. We also fix some $\varepsilon > 0$. Henceforth in this chapter, the multiplicative constant in expressions of the form $O(f(n))$ may depend on ε.

Recall the definition (2.4). Make a change of variable $r := R + it + \varepsilon$ and denote $\mathcal{R}(t) = \{r : r = it + \varepsilon + \rho, \rho \geq 0\}$. Notice that the integrand is even with respect to u_2, thus we can leave the integration with respect to u_2 over

$[0, +\infty)$ and write a multiplier 2 before the integral. Now observe that

$$L_n(u_1^2+u_2^2)-(u_1+\varepsilon)^2-u_2^2 = L_n(u_1^2+u_2^2)-\left(\sqrt{u_1^2+u_2^2}-\varepsilon\right)^2-2\varepsilon\left(u_1+\sqrt{u_1^2+u_2^2}\right).$$

Since $u_1 + \sqrt{u_1^2+u_2^2} \geq 0$, we can make a change $(u_1, u_2) \to (u, w)$, $u = \sqrt{u_1^2+u_2^2} \in [0, +\infty)$, $w = \sqrt{u_1+u} \in [0, \sqrt{2u}]$. One can see that $u_1 = w^2 - u$, $u_2 = \sqrt{u^2 - u_1^2}$ and the Jacobian of this change is equal to $J = -\dfrac{2u}{\sqrt{2u-w^2}}$.

Set $\mathbf{x} = (u, s, t, r, v, w)$ and $\mathcal{V} = [0, +\infty) \times \mathbb{R} \times L_t \times \mathcal{R}(t) \times \mathbb{R} \times [0, \sqrt{2u}]$. Then

$$\mathbf{I}_{g_n} = \frac{8n^3}{\pi^3} \int_{\mathcal{V}} \Phi_n(\mathbf{x})\, e^{nF_n(\mathbf{x})}\, d\mathbf{x},$$

where

$$F_{n,1}(u) = \mathcal{L}_n(u^2) - (u-\varepsilon)^2, \quad F_{n,2}(s,t,r) = -\Big(\mathcal{L}_n(s^2-t^2) + (t-i\varepsilon)^2 + r^2\Big),$$

$$F_n(\mathbf{x}) = F_{n,1}(u) + F_{n,2}(s,t,r) - 2\varepsilon w^2 - \varepsilon v^2,$$

$$\Phi_n(\mathbf{x}) = \frac{r - it - \varepsilon}{\sqrt{v^2 + 4r - 4it - 4\varepsilon}} \cdot \frac{u}{\sqrt{2u - w^2}} \cdot g_n(u^2, s^2 - t^2).$$

Step 1: analysis of $F_{n,1}(u)$

Let us find the maximum point of the function $F_{n,1}(u)$ for $u \in [0, +\infty)$. Obviously, $\lim\limits_{u\to+\infty} F_{n,1}(u) = -\infty$, hence the maximum point $u' \geq 0$ exists and satisfies $\partial_u F_{n,1}(u') = 0$. Straightforward computation gives us

$$\partial_u F_{n,1}(u) = 2u \cdot \mathrm{tr}_n G(u^2) - 2u + 2\varepsilon.$$

Then $u' = x_{\varepsilon,n}$, where $x_{\varepsilon,n}$ is the positive solution of (3.1) defined in Section 3.1. We can expand $F_{n,1}(u)$ for u lying in some neighbourhood of $x_{\varepsilon,n}$:

$$F_{n,1}(u) = \mathcal{L}_n(x_{\varepsilon,n}^2) - (x_{\varepsilon,n} - \varepsilon)^2 - \kappa_1 (u - x_{\varepsilon,n})^2 + O\big(|u - x_{\varepsilon,n}|^3\big), \qquad (3.6)$$

where

$$\kappa_1 = \frac{\varepsilon}{x_{\varepsilon,n}} + \mathrm{tr}_n G^2(x_{\varepsilon,n}^2) \cdot 2x_{\varepsilon,n}^2. \tag{3.7}$$

Proposition 3.2 implies that $\kappa_1 \geq C > 0$ uniformly, thus

$$F_{n,1}(u) \leq F_{n,1}(x_{\varepsilon,n}) - Cn^{-1}\log^2 n \quad \text{when } |u - x_{\varepsilon,n}| > n^{-1/2}\log n, \tag{3.8}$$

for large n and small ε, where $C > 0$ does not depend on n, ε.

Step 2: analysis of $F_{n,2}(s,t,r)$.

Consider the function $h_n(t) = -\mathcal{L}_n(-t^2) - (t - i\varepsilon)^2$. It is easy to see that $h_n(t)$ is analytic in the upper halfplane and $t = ix_{\varepsilon,n}$ is a stationary point of $h_n(t)$.

Lemma 3.6. *There exist constants δ, σ, C_0 and a contour L_1 which connects points $ix_{\varepsilon,n} + \delta$ and $C_0 + i(C_0 + \varepsilon)$ satisfying the following conditions:*

- $0 < \delta < \dfrac{\kappa_0}{2}$, $\sigma > 0$, $C_0 > 0$ *and* $\log 2C_0^2 > \mathcal{L}_n(x_{\varepsilon,n}^2) + 1$;
- δ, σ, C_0 *do not depend on* ε, n;
- $L_1 \subset \{z \mid \Re z > 0, \Im z - \Re z \geq \varepsilon\}$;
- $\Re h_n(ix_{\varepsilon,n} + t)$ *decreases for* $t \in [0, \delta]$, *and* $\Re h_n(t) \leq \Re h_n(ix_{\varepsilon,n}) - \sigma$ *for* $t \in L_1$.

Proof. This lemma is similar to [23, Lemma 4.1].

First one can prove that there exist $\delta, \sigma > 0$ independent of n, ε such that $\Re h_n(ix_{\varepsilon,n} + t)$ decreases for $t \in [0, \delta]$ and $\Re h_n(ix_{\varepsilon,n} + \delta) \leq \Re h_n(ix_{\varepsilon,n}) - \sigma$. Set $\widetilde{h}_n(t) = \Re h_n(ix_{\varepsilon,n} + t)$, then it follows from the fact that $\widetilde{h}_n'(0) = 0$, $\widetilde{h}_n''(0) = -2\kappa_1$, where κ_1 defined in (3.7) is bounded from below by some positive constant, and $\widetilde{h}_n'''(t)$ is bounded uniformly in ε, n for small t.

We proceed to a construction of the contour L_1. Recall that $h_n(t)$ is an analytic function in the upper halfplane. We are going to study the level line ℓ of $\Re h_n(t)$ which passes through $ix_{\varepsilon,n} + \delta$.

Let $\ell_1 = \{i\tau \mid \tau > 0\}$. One can see that $ix_{\varepsilon,n}$ is the minimum point of $\Re h_n(t)$ on ℓ_1. The inequality $\Re h_n(ix_{\varepsilon,n} + \delta) < \Re h_n(ix_{\varepsilon,n} + t) < \Re h_n(ix_{\varepsilon,n})$, $t \in [0, \delta]$ implies that the level line ℓ cannot intersect ℓ_1 and $[ix_{\varepsilon,n},\, ix_{\varepsilon,n} + \delta]$.

Set $\ell_2 = \{t(1 + i) + i\varepsilon,\ t > 0\}$, $\ell_3 = \{iM_1 + t, t > 0\}$, where $M_1 = M + |z|$ and M is defined in (C2). Similarly to [23, Lemma 4.1] one can show that $\Re h(t)$ decreases on the contours ℓ_2 and ℓ_3. It means that ℓ can intersect each of ℓ_2 and ℓ_3 at most once. The fact that ℓ cannot intersect ℓ_1 implies that ℓ intersect each of ℓ_2 and ℓ_3 exactly once.

Consider the part ℓ' of ℓ which connects $ix_{\varepsilon,n}$ with some point on ℓ_2, say $t_0(1 + i) + i\varepsilon$. If $\log 2t_0^2 > \mathcal{L}_n(x_{\varepsilon,n}^2) + 1$ then choose ℓ' as L_1. Otherwise, take some $C_0 > 0$ such that $\log 2C_0^2 > \mathcal{L}_n(x_{\varepsilon,n}^2) + 1$ and choose $L_1 = \ell' \cup [t_0(1 + i) + i\varepsilon,\, C_0(1 + i) + i\varepsilon]$. In both cases the inequality $\Re h(t) \le \Re h(ix_{\varepsilon,n} + \delta) \le \Re h(ix_{\varepsilon,n}) - \sigma$ holds for $t \in L_1$.

$\square$

Take L_2 to be symmetric to L_1 with respect to the imaginary axis, and

$$L_0 := [ix_{\varepsilon,n} - \delta, ix_{\varepsilon,n} + \delta], \quad L'_{1,2} := \{i(C_0 + \varepsilon) \pm C_0 \pm \tau, \tau > 0\}$$

Then we can move the integration with respect to t to the contour $L_t := L_0 \cup L_1 \cup L_2 \cup L'_1 \cup L'_2$. Also we deform the r-contour for each $t \in L_t$ as follows: $\widetilde{\mathcal{R}}(t) = \mathcal{R}_1(t) \cup \mathcal{R}_2(t)$, where $\mathcal{R}_1(t) = [it + \varepsilon, -\delta]$, $\quad \mathcal{R}_2(t) = \{r\colon r = -\delta + \rho, \rho \ge 0\}$. Next we prove that $(0, ix_{\varepsilon,n}, 0)$ is a 'good enough' maximum point of $\Re F_{n,2}(s, t, r)$ when $s \in \mathbb{R}$, $t \in L_t$, $r \in \widetilde{\mathcal{R}}(t)$. More precisely:

Proposition 3.7. $(0, ix_{\varepsilon,n}, 0)$ *is the maximum point of* $\Re F_{n,2}(s, t, r)$ *when*

$s \in \mathbb{R}$, $t \in L_t$ *and* $r \in \widetilde{\mathcal{R}}(t)$. *Moreover,*

$$\begin{aligned}\Re F_{n,2}(s,t,r) \leq \Re F_{n,2}(0, ix_{\varepsilon,n}, 0) - Cn^{-1}\log^2 n \quad \text{when} \\ \max\{|s|, |t - ix_{\varepsilon,n}|, |r|\} > n^{-1/2}\log n.\end{aligned} \tag{3.9}$$

Proof. The statement above is a straightforward consequence of the following inequalities:

$$\Re F_{n,2}(s,t,r) \leq \Re F_{n,2}(0, ix_{\varepsilon,n}, 0) - 1 \quad \text{when } t \in L'_{1,2}; \tag{3.10}$$

$$\Re F_{n,2}(s,t,r) \leq \Re F_{n,2}(s,t,0) \quad \text{when } t \in L_{0,1,2}; \tag{3.11}$$

$$\Re F_{n,2}(s,t,0) \leq \Re F_{n,2}(0,t,0) \quad \text{when } t \in L_{0,1,2}; \tag{3.12}$$

$$\Re F_{n,2}(0,t,0) \leq \Re F_{n,2}(0, ix_{\varepsilon,n}, 0) - \sigma \quad \text{when } t \in L_{1,2}; \tag{3.13}$$

$$\Re F_{n,2}(0,t,0) \leq \Re F_{n,2}(0, ix_{\varepsilon,n}, 0) \quad \text{when } t \in L_0; \tag{3.14}$$

$$\Re F_{n,2}(s,t,0) \leq \Re F_{n,2}(0,t,0) - C\epsilon^2 \text{ when } t \in L_0,\ |t - ix_{\varepsilon,n}| < \epsilon,\ |s| > \epsilon; \tag{3.15}$$

$$\Re F_{n,2}(0,t,0) \leq \Re F_{n,2}(0, ix_{\varepsilon,n}, 0) - C\epsilon^2 \text{ when } t \in L_0,\ |t - ix_{\varepsilon,n}| > \epsilon; \tag{3.16}$$

$$\Re F_{n,2}(s,t,r) \leq \Re F_{n,2}(s,t,0) - C\epsilon^2 \quad \text{when } t \in L_0,\ |r| > \epsilon. \tag{3.17}$$

Let us start from the proof of (3.10). Notice that $\Re F_{n,2}(0, ix_{\varepsilon,n}, 0) = -\mathcal{L}_n(x^2_{\varepsilon,n}) + (x_{\varepsilon,n} - \varepsilon)^2 \geq -\mathcal{L}_n(x^2_{\varepsilon,n})$. Observe that for $t \in L'_{1,2}$ we have

$$\begin{aligned}-\Re\mathcal{L}_n(s^2 - t^2) &= -\Re\mathcal{L}_n(s^2 + (C_0+\varepsilon)^2 - (C_0+\tau)^2 \pm 2i(C_0+\varepsilon)(C_0+\tau)) = \\ &= -\frac{1}{2}\int \log\Big((\lambda + s^2 + (C_0+\varepsilon)^2 - (C_0+\tau)^2)^2 + \\ &+ (2(C_0+\varepsilon)(C_0+\tau))^2\Big) d\nu_{n,z}(\lambda) \leq -\log 2C_0^2 \leq -\mathcal{L}_n(x^2_{\varepsilon,n}) - 1\end{aligned} \tag{3.18}$$

Also we have $\Re(-r^2 - (t - i\varepsilon)^2) \leq 0$. Indeed, for $r \in \mathcal{R}_1(t)$ we have $r =$

$\alpha(it+\varepsilon)-(1-\alpha)\delta$, $\alpha\in[0;1]$ and $t=C_0+\tau+i(C_0+\varepsilon)$, thus

$$\Re(-r^2-(t-i\varepsilon)^2)=-(\alpha C_0+(1-\alpha)\delta)^2+\alpha^2(C_0+\tau)^2-(C_0+\tau)^2+C_0^2\le \\ \le(1-\alpha^2)(C_0^2-(C_0+\tau)^2)\le 0;$$

and for $r\in\mathcal{R}_2(t)$, $r=-\delta+\tau$, we have $\Re(-r^2)=-r^2\le 0$, $\Re(-(t-i\varepsilon)^2)=C_0^2-(C_0+\tau)^2\le 0$. Now (3.10) follows from (3.18) and $\Re(-r^2-(t-i\varepsilon)^2)\le 0$.

Next, we prove (3.11). It suffices to prove that $\Re r^2\ge 0$. Set $t=t_1+it_2$, then $t_2-|t_1|\ge\varepsilon>0$ for $t\in L_{0,1,2}$. For $r\in\mathcal{R}_2(t)$ we have $r\in\mathbb{R}$, thus $\Re r^2=r^2\ge 0$. For $r\in\mathcal{R}_1(t)$ we have $r=\alpha(-t_2+\varepsilon+it_1)-(1-\alpha)\delta$, thus

$$\Re r^2=(-\alpha(t_2-\varepsilon)-(1-\alpha)\delta)^2-(\alpha t_1)^2\ge\alpha^2((t_2-\varepsilon)^2-t_1^2)\ge 0.$$

Next, we prove (3.12). Set $t=t_1+it_2$, then $t_2-|t_1|\ge\varepsilon>0$ since $t\in L_{0,1,2}$. Also, $t^2=t_1^2-t_2^2+2it_1t_2$, thus

$$\Re F_{n,2}(s,t,0)=-\frac{1}{2}\int\log((\lambda+t_2^2-t_1^2+s^2)^2+4t_1^2t_2^2)\,d\nu_{n,z}(\lambda)-\Re(t-i\varepsilon)^2$$

Since $t_2^2-t_1^2>0$, then $\Re F_{n,2}(s,t,0)$ decreases for $s\in[0,+\infty)$, which gives us (3.12).

Notice that $\Re F_{n,2}(0,t,0)=\Re h_n(t)$. The inequalities $\Re h_n(t)\le\Re h_n(ix_{\varepsilon,n})-\sigma$ for $t\in L_{1,2}$ and $\Re h_n(t)\le\Re h_n(ix_{\varepsilon,n})$ for $t\in L_0$ imply that (3.13) and (3.14) hold.

The inequality (3.15) follows from the fact that $\Re F_{n,2}(s,t,0)$ decreases for $s\in[0,+\infty)$ and $\partial_s\partial_s\Re F_{n,2}(s,t,0)\big|_{s=0}=-2\Re\mathrm{tr}_nG(-t^2)<-1$ for t lying in some neighbourhood of $ix_{\varepsilon,n}$, while (3.16) follows from the fact that $\Re F_{n,2}(0,t,0)=\Re h_n(t)$ decreases when $t\in[ix_{\varepsilon,n},ix_{\varepsilon,n}+\delta]$, and $\partial_\tau\partial_\tau\Re h_n(ix_{\varepsilon,n}+\tau)\big|_{\tau=0}=-2\kappa_1$.

Finally, we prove (3.17). It suffices to show that $\Re r^2\ge C\epsilon^2$. Set $t=ix_{\varepsilon,n}+\tau$, $|\tau|<\frac{1}{2}x_{\varepsilon,n}$. In case $r\in\mathcal{R}_1(t)$ we have

$r = \alpha(-x_{\varepsilon,n} + i\tau + \varepsilon) - (1-\alpha)\delta$, and

$$\Re r^2 = (-\alpha(x_{\varepsilon,n} - \varepsilon) - (1-\alpha)\delta)^2 - (\alpha\tau)^2 \geq \alpha^2((x_{\varepsilon,n} - \varepsilon)^2 - \tau^2) + \\ + (1-\alpha)^2\delta^2 \geq C > 0,$$

for small τ. In case $r \in \mathcal{R}_2(t)$, we have $\Re r^2 = r^2 = |r|^2 \geq \epsilon^2$.

$\square$

The proposition above will allow us to restrict the integration with respect to s, t, r to the set

$$U_{2,n} = \{(s,t,r) \in \mathbb{R} \times \mathcal{L}_t \times \widetilde{\mathcal{R}}(t) \colon |s|, |t - ix_{\varepsilon,n}|, |r| < n^{-1/2} \log n\}.$$

It is easy to check that for (s,t,r) lying in some neighbourhood of $(0, ix_{\varepsilon,n}, 0)$ we have

$$F_{n,2}(s,t,r) = -\mathcal{L}_n(x_{\varepsilon,n}^2) + (x_{\varepsilon,n} - \varepsilon)^2 - \kappa_1(t - ix_{\varepsilon,n})^2 - \kappa_3 s^2 - r^2 + \\ + O(|s|^3 + |t - ix_{\varepsilon,n}|^3), \tag{3.19}$$

where κ_1 defined in (3.7) and

$$\kappa_3 = \mathrm{tr}_n G(x_{\varepsilon,n}^2). \tag{3.20}$$

Proposition 3.2 implies that $\kappa_3 \geq C > 0$ uniformly.

Step 3: restricting the integration domain

Observe that (3.8) and (3.9) together with $F_{n,1}(x_{\varepsilon,n}) = -F_{n,2}(0, ix_{\varepsilon,n}, 0)$ give us $\Re F_n(\mathbf{x}) \leq -Cn^{-1}\log^2 n$ when $\max\{|u - x_{\varepsilon,n}|, |s|, |t - ix_{\varepsilon,n}|, |r|, |v|, |w|\} > n^{-1/2}\log n$. Thus we can restrict the integration to the set

$$U_n = \{\mathbf{x} \in \widetilde{\mathcal{V}} \colon |u - x_{\varepsilon,n}|,\ |s|,\ |t - ix_{\varepsilon,n}|,\ |r|,\ |v|,\ |w| < n^{-1/2}\log n\}.$$

Make the changes of variables $u = x_{\varepsilon,n} + n^{-1/2}\widetilde{u}$, $s = n^{-1/2}\widetilde{s}$, $t = ix_{\varepsilon,n} + n^{-1/2}\widetilde{t}$, $r = n^{-1/2}\widetilde{r}$, $v = n^{-1/2}\widetilde{v}$, $w = n^{-1/2}\widetilde{w}$, use the expansions (3.6), (3.19), and expand the integral, then

$$\begin{aligned}\mathbf{I}_{g_n} &= g_n(x_{\varepsilon,n}^2, x_{\varepsilon,n}^2) \cdot \frac{x_{\varepsilon,n} - \varepsilon}{\sqrt{4x_{\varepsilon,n} - 4\varepsilon}} \cdot \frac{x_{\varepsilon,n}}{\sqrt{2x_{\varepsilon,n}}} \times \\ &\times \frac{8}{\pi^3} \int_{\widetilde{U}_n} e^{-\kappa_1\widetilde{u}^2 - \kappa_1\widetilde{t}^2 - \kappa_3\widetilde{s}^2 - \widetilde{r}^2 - \varepsilon\widetilde{v}^2 - 2\varepsilon\widetilde{w}^2}\, d\widetilde{u}\, d\widetilde{t}\, d\widetilde{s}\, d\widetilde{r}\, d\widetilde{v}\, d\widetilde{w} + O(n^{-1/2}\log^k n),\end{aligned}$$

where $\widetilde{U}_n = \{(\widetilde{u}, \widetilde{s}, \widetilde{t}, \widetilde{r}, \widetilde{v}, \widetilde{w}) \in \mathbb{R}^6 \colon |\widetilde{u}|, |\widetilde{s}|, |\widetilde{t}|, |\widetilde{r}|, |\widetilde{v}| < \log n,\ 0 \le \widetilde{w} < \log n\}$. Notice that $e^{-\kappa_1\widetilde{u}^2 - \kappa_1\widetilde{t}^2 - \kappa_3\widetilde{s}^2 - \widetilde{r}^2 - \varepsilon\widetilde{v}^2 - 2\varepsilon\widetilde{w}^2} < e^{-C\log^2 n}$ for $(\widetilde{u}, \widetilde{s}, \widetilde{t}, \widetilde{r}, \widetilde{v}, \widetilde{w}) \in \mathbb{R}^6 \backslash \widetilde{U}_n$, thus the integration over $\mathbb{R}^6 \setminus \widetilde{U}_n$ contributes as $O(e^{-C\log^2 n})$. This argument allows us to integrate over $\mathbb{R}^6$. We get

$$\mathbf{I}_{g_n} = C(\varepsilon, n, z) \cdot g_n(x_{\varepsilon,n}^2, x_{\varepsilon,n}^2) + O(n^{-1/3}), \tag{3.21}$$

where $C(\varepsilon, n, z) = \sqrt{\dfrac{(x_{\varepsilon,n} - \varepsilon)x_{\varepsilon,n}}{\kappa_1^2\kappa_3\varepsilon^2}}$. Notice that $C(\varepsilon, n, z)$ does not depend on g_n and $C(\varepsilon, n, z) = O(1)$ for fixed $\varepsilon > 0$ as $n \to \infty$. Substituting $g_n(u_1^2 + u_2^2, s^2 - t^2) = \varphi_n(u_1^2 + u_2^2, s^2 - t^2, z, z)$ in (3.21), we obtain

$$\mathbf{I}_{\varphi_n(z,z)} = C(\varepsilon, n, z) \cdot \varphi_n(x_{\varepsilon,n}^2, x_{\varepsilon,n}^2, z, z) + O(n^{-1/3})$$

On the other hand, according to (2.5),

$$\mathbf{I}_{\varphi_n(z,z)} = \Big\langle \varphi_n(u_1^2 + u_2^2, s^2 - t^2, z, z) e^{n\mathcal{F}_n(z,z)} \Big\rangle = \mathcal{Z}(\varepsilon, \varepsilon, z, z) = 1.$$

Therefore,

$$\mathbf{I}_{g_n} = \frac{\mathbf{I}_{g_n}}{\mathbf{I}_{\varphi_n(z,z)}} = \frac{g_n(x_{\varepsilon,n}^2, x_{\varepsilon,n}^2)}{\varphi_n(x_{\varepsilon,n}^2, x_{\varepsilon,n}^2, z, z)} + O(n^{-1/3}),$$

which gives us (3.5) in case $z \in \operatorname{Int} D$.

3.3. Integrals of the form $\left\langle g \cdot e^{n\mathcal{F}_n(z,z)} \right\rangle$, case $z \in \mathbb{C} \setminus \overline{D}$

We derive the same asymptotic formula for $\mathbf{I}_{g_n} = \left\langle g_n(u_1, u_2, t, s) e^{n\mathcal{F}_n(z,z)} \right\rangle$ as in the previous section, but for $z \in \mathbb{C} \setminus \overline{D}$. Recall (2.4), make a change of variable $r := R + it + \varepsilon$ and denote $\mathcal{R}(t) = \{r \colon r = it + \varepsilon + \rho, \rho \geq 0\}$. Then

$$\mathbf{I}_{g_n} = \frac{2n^3}{\pi^3} \int_{\mathcal{V}} \Phi_n(\mathbf{x}) e^{nF_n(\mathbf{x})} \, d\mathbf{x},$$

where

$$\begin{aligned}
&\mathbf{x} = (u_1, u_2, s, t, r, v), \quad \mathcal{V} = \mathbb{R}^2 \times \mathbb{R} \times L_t \times \mathcal{R}(t) \times \mathbb{R}, \\
&F_{n,1}(u_1, u_2) = \mathcal{L}_n(u_1^2 + u_2^2) - (u_1 + \varepsilon)^2 - u_2^2, \\
&F_{n,2}(s, t, r) = -\Big(\mathcal{L}_n(s^2 - t^2) + (t - i\varepsilon)^2 + r^2\Big), \\
&F_n(\mathbf{x}) = F_{n,1}(u_1, u_2) + F_{n,2}(s, t, r) - \varepsilon v^2, \\
&\Phi_n(\mathbf{x}) = \frac{r - it - \varepsilon}{\sqrt{v^2 + 4r - 4it - 4\varepsilon}} \cdot g_n(u_1, u_2, t, s).
\end{aligned}$$

Fix some $z \in \mathbb{C} \setminus \overline{D}$. We will use the results of Section 3.1 for $E_{out} = \{z\}$.

First we study $F_{n,1}(u_1, u_2)$. The maximum point (u_1', u_2') of $F_{n,1}$ must be a solution of the following system:

$$\partial_{u_1} F_{n,1}(u_1', u_2') = \partial_{u_2} F_{n,1}(u_1', u_2') = 0,$$

which is equivalent to $u_2' = 0$ and $(-u_1)'$ is a solution of (3.1). According to Proposition 3.3, the equation (3.1) has exactly one positive root $x_{\varepsilon,n}$ and no negative roots, thus $(u_1', u_2') = (-x_{\varepsilon,n}, 0)$ is the maximum point of $F_{n,1}(u_1, u_2)$.

Now we can obtain the same expansion for $F_{n,1}$ in the neighbourhood of

$(-x_{\varepsilon,n}, 0)$ as in previous section:

$$F_{n,1}(u_1, u_2) = \mathcal{L}_n(x_{\varepsilon,n}^2) - (x_{\varepsilon,n} - \varepsilon)^2 - \kappa_1(u + x_{\varepsilon,n})^2 - \kappa_2 u_2^2 + O(|u_1 + x_{\varepsilon,n}|^3 + |u_2|^3),$$

where κ_1 defined in (3.7) and

$$\kappa_2 = \frac{\varepsilon}{x_{\varepsilon,n}}. \tag{3.22}$$

Proposition 3.4 gives us $\kappa_{1,2} \geq C > 0$, hence

$$F_{n,1}(u_1, u_2) \leq F_{n,1}(-x_{\varepsilon,n}, 0) - Cn^{-1}\log^2 n \quad \text{when} \\ \max\{|u_1 + x_{\varepsilon,n}|, |u_2|\} > n^{-1/2}\log n,$$

for large n and small ε, where $C > 0$ does not depend on ε and n.

Next we study $F_{n,2}(s, t, r)$. Again, consider $h_n(t) = -\mathcal{L}_n(-t^2) - (t - i\varepsilon)^2$. First we prove a lemma similar to Lemma 3.6 in the case $z \in \mathbb{C} \setminus \overline{D}$.

Lemma 3.8. *There exist constants δ, σ, C_0 and a contour $\mathbf{L}_1$ which connects points $ix_{\varepsilon,n} + \delta(1 + i(1 + \varepsilon - x_{\varepsilon,n}))$ and $C_0 + i(C_0 + \varepsilon)$ satisfying the following conditions:*

- $0 < \delta < 1$, $\sigma > 0$, $C_0 > 0$ *and* $\log 2C_0^2 > \mathcal{L}_n(x_{\varepsilon,n}^2) + 1$*;*
- δ, σ, C_0 *do not depend on* ε, n*;*
- $\mathbf{L}_1 \subset \{z \mid \Re z > 0, \Im z - \Re z \geq \varepsilon\}$*;*
- $\Re h_n\big(ix_{\varepsilon,n} + t(1 + i(1 + \varepsilon - x_{\varepsilon,n}))\big)$ *decreases for* $t \in [0, \delta]$*, and* $\Re h_n(t) \leq \Re h_n(ix_{\varepsilon,n}) - \sigma$ *for* $t \in \mathbf{L}_1$*.*

Proof.

Set $\widetilde{h}_n(\tau)=\Re h_n(ix_{\varepsilon,n}+\tau(1+i(1+\varepsilon-x_{\varepsilon,n})))$. One can check that $\widetilde{h}_n'(0)=0$,

$$\widetilde{h}_n''(0)=-\frac{4\varepsilon}{x_{\varepsilon,n}}(x_{\varepsilon,n}-\varepsilon)+O(\varepsilon^2),\quad \widetilde{h}_n'''(0)=-24x_{\varepsilon,n}\cdot\mathrm{tr}_nG^2(x_{\varepsilon,n}^2)+O(\varepsilon^2),$$
$$\widetilde{h}_n^{(IV)}(0)=-48\cdot\mathrm{tr}_nG^2(x_{\varepsilon,n}^2)+O(\varepsilon),\ \widetilde{h}_n^{(V)}(0)=768x_{\varepsilon,n}\cdot\mathrm{tr}_nG^3(x_{\varepsilon,n}^2)+O(\varepsilon^2).$$

Observe that $\widetilde{h}_n''(0)<0$, $\widetilde{h}_n'''(0)<0$ for small ε, $|\widetilde{h}_n^{(V)}(0)|$ is bounded and $-\widetilde{h}_n^{(IV)}(0)\gg 1$ as $\varepsilon\to 0$, $n\to\infty$. This shows that there exist $\delta,\sigma>0$ independent of n,ε such that $\widetilde{h}_n(\tau)$ decreases for $\tau\in[0,\delta]$ and $\widetilde{h}_n(\delta)\le\widetilde{h}_n(0)-\sigma$.

The rest of the proof is the same as in Lemma 3.6. □

Now we change the t-contour and r-contour as follows. Let $\mathbf{L}_2$ be a contour symmetric to $\mathbf{L}_1$ with respect to imaginary axis, and set

$$\mathbf{L}_0^{\pm}:=[ix_{\varepsilon,n},ix_{\varepsilon,n}+\delta\big(\pm1+i(1+\varepsilon-x_{\varepsilon,n})\big)],\quad \mathbf{L}_{1,2}':=\{(i(C_0+\varepsilon)\pm C_0\pm\tau,\tau>0\}$$

Then we can move the integration with respect to t to the following contour: $\mathbf{L}_t:=\mathbf{L}_0^+\cup\mathbf{L}_0^-\cup\mathbf{L}_1\cup\mathbf{L}_2\cup\mathbf{L}_1'\cup\mathbf{L}_2'$. Also we can move the integration with respect to r to the contour $\widetilde{\mathcal{R}}(t)=\mathcal{R}_1(t)\cup\mathcal{R}_2(t)$, where $\mathcal{R}_1(t)=[it+\varepsilon,-\delta]$, $\mathcal{R}_2(t)=\{r\colon r=-\delta+\rho,\rho\ge 0\}$. One can show that for (s,t,r) lying in some neighbourhood of $(0,ix_{\varepsilon,n},0)$, $t=ix_{\varepsilon,n}+\tau(\pm1+i(1+\varepsilon-x_{\varepsilon,n}))$ we have

$$F_{n,2}(s,t,r)=-\mathcal{L}_n(x_{\varepsilon,n}^2)+(x_{\varepsilon,n}-\varepsilon)^2-(\kappa_3s^2+\kappa_4\tau^2+r^2)+O(|s|^3+\tau^3),$$

where κ_1, κ_3 are defined in (3.7), (3.20) and

$$\kappa_4=\Big(\frac{\varepsilon}{x_{\varepsilon,n}}+2x_{\varepsilon,n}^2\cdot\mathrm{tr}_nG^2(x_{\varepsilon,n}^2)\Big)\cdot(\pm2i+2(1\mp i)(x_{\varepsilon,n}-\varepsilon)+(x_{\varepsilon,n}-\varepsilon)^2).\quad(3.23)$$

One can check that $\Re\kappa_4\ge C\varepsilon$ for small ε and some $C>0$ independent of n,ε. Using the expansion above and the estimations similar to the ones from the previous section, one can show that

$$\begin{aligned}\Re F_{n,2}(s,t,r) \le \Re F_{n,2}(0, ix_{\varepsilon,n}, 0) - C\varepsilon n^{-1}\log^2 n \quad \text{when}\\ \max\{|s|,|r|,|t-ix_{\varepsilon,n}|\} > n^{-1/2}\log n.\end{aligned} \tag{3.24}$$

Now consider the set

$$U_n = \{\mathbf{x} \in \widetilde{\mathcal{V}} \colon |u_1 + x_{\varepsilon,n}|,\ |u_2|,\ |s|,\ |t - ix_{\varepsilon,n}|,\ |r|,\ |v| < n^{-1/2}\log n\}.$$

The estimations above show that $\Re F_n(\mathbf{x}) \le Cn^{-1}\log^2 n$ for $\mathbf{x} \in \widetilde{\mathcal{V}} \setminus U_n$ and a fixed $\varepsilon > 0$, which allows us restrict the integration to U_n. Make the following change of variables:

$$\begin{aligned}&u_1 = n^{-1/2}\widetilde{u}_1, \quad u_2 = n^{-1/2}\widetilde{u}_2, \quad s = n^{-1/2}\widetilde{s}, \quad r = n^{-1/2}\widetilde{r}, \quad v = n^{-1/2}\widetilde{v},\\ &t = ix_{\varepsilon,n} + n^{-1/2}\widetilde{t}_\pm(\pm 1 + i(1+\varepsilon - x_{\varepsilon,n})) \quad \text{for } t \in L_0^\pm.\end{aligned}$$

Set $\widetilde{U}_n = \{(\widetilde{u}_1, \widetilde{u}_2, \widetilde{s}, \widetilde{t}, \widetilde{r}, \widetilde{v}) \in \mathbb{R}^6 \colon |\widetilde{u}_1|, |\widetilde{u}_2|, |\widetilde{s}|, |\widetilde{r}|, |v| < \log n, 0 \le \widetilde{t}_\pm < \log n\}$. Expanding the integrand, we obtain

$$\begin{aligned}\mathbf{I}_{g_n} =& \frac{x_{\varepsilon,n} - \varepsilon}{\sqrt{4(x_{\varepsilon,n} - \varepsilon)}} \cdot g_n(x_{\varepsilon,n}^2, x_{\varepsilon,n}^2) \cdot \frac{2}{\pi^3} \int_{\widetilde{U}_n} d\widetilde{u}_1\, d\widetilde{u}_2\, d\widetilde{t}_\pm\, d\widetilde{s}\, d\widetilde{r}\, d\widetilde{v} \times\\ &\times \exp\{-\kappa_1 \widetilde{u}_1^2 - \widetilde{\kappa}_2 \widetilde{u}_2^2 - \kappa_3 \widetilde{s}^2 - \kappa_4 \widetilde{t}_\pm^2 - \widetilde{r}^2 - \varepsilon \widetilde{v}^2\} +\\ &+ O(n^{-1/2}\log^k n) = C(\varepsilon, n, z) \cdot g_n(x_{\varepsilon,n}^2, x_{\varepsilon,n}^2) + O(n^{-1/3}),\end{aligned}$$

where $C(\varepsilon, n, z)$ does not depend on g_n. Again, $C(\varepsilon, n, z) = O(1)$ as $n \to \infty$ for some fixed $\varepsilon > 0$. Taking $g_n(x,y) = \varphi_n(x,y,z,z)$ and using the relation $\mathcal{Z}(\varepsilon, \varepsilon, z, z) = 1$ as in the previous section, we obtain

$$\mathbf{I}_{g_n} = \frac{g_n(x_{\varepsilon,n}^2, x_{\varepsilon,n}^2)}{\varphi_n(x_{\varepsilon,n}^2, x_{\varepsilon,n}^2, z, z)} + O(n^{-1/3}),$$

which gives us (3.5) for $z \in \mathbb{C} \setminus \overline{D}$.

3.4. Asymptotic behaviour of $\mathbf{I}_k$ and $\rho_{\varepsilon,n,\omega}(z)$

Using the formula (3.5), we can now easily obtain the following result.

Proposition 3.9. *For $\mathbf{I}_2, \mathbf{I}_3, \mathbf{I}_4$ defined in (2.7) and $z \notin \partial D$ we have*

$$\mathbf{I}_2 = x_{\varepsilon,n}^2 \cdot \mathrm{tr}_n G(x_{\varepsilon,n}^2)\widetilde{G}(x_{\varepsilon,n}^2) + O(n^{-\alpha}), \quad \mathbf{I}_3 = O(n^{-\alpha}), \quad \mathbf{I}_4 = O(n^{-1}). \tag{3.25}$$

Proof. Take $g_{n,2} = \Big((\partial_{\bar z} + \partial_{\bar z_1})\partial_{z_1}\mathcal{F}_n(z,z_1) \cdot \varphi_n(z,z_1)\Big)\Big|_{z_1=z}$. One can show that

$$g_{n,2}(u_1^2+u_2^2, s^2-t^2) = (u_1^2+u_2^2)\cdot \mathrm{tr}_n G(u_1^2+u_2^2)\widetilde{G}(u_1^2+u_2^2)\times \\ \times \varphi_n(u_1^2+u_2^2, s^2-t^2, z, z).$$

Applying (3.5), we obtain

$$\mathbf{I}_2 = \mathbf{I}_{g_{n,2}} = x_{\varepsilon,n}^2 \cdot \mathrm{tr}_n G(x_{\varepsilon,n}^2)\widetilde{G}(x_{\varepsilon,n}^2) + O(n^{-\alpha})$$

Next, we take $g_{n,3} = \Big(\partial_{z_1}\varphi_n(z,z_1)\cdot(\partial_{\bar z} + \partial_{\bar z_1})\mathcal{F}_n(z,z_1)\Big)\Big|_{z_1=z}$. Note that

$$\Big((\partial_{\bar z} + \partial_{\bar z_1})\mathcal{F}_n(z,z_1)\Big)\Big|_{z_1=z} = \mathrm{tr}_n\,(z-A_n)G(u_1^2+u_2^2) - \mathrm{tr}_n\,(z-A_n)G(s^2-t^2),$$

thus $g_{n,3}(x_{\varepsilon,n}^2, x_{\varepsilon,n}^2) = 0$. Applying (3.5) we get $\mathbf{I}_3 = \mathbf{I}_{g_{n,3}} = O(n^{-\alpha})$. Finally, take $g_{n,4} = \Big((\partial_{\bar z} + \partial_{\bar z_1})\partial_{z_1}\varphi_n(z,z_1)\Big)\Big|_{z_1=z}$. Then $\mathbf{I}_4 = \dfrac{1}{n}\mathbf{I}_{g_{n,4}} = O(n^{-1})$. □

Next, we are going to find an asymptotic formula for $\mathbf{I}_1$. If we set

$$g_{n,1} = \partial_{z_1}\mathcal{F}_n(z,z)\Big(\partial_{\bar z} + \partial_{\bar z_1}\Big)\Big(\varphi_n(z,z_1)\,e^{n\mathcal{F}_n(z,z_1)}\Big)\cdot e^{-n\mathcal{F}_n(z,z)}$$

then we cannot apply (3.5) for $g_n = g_{n,1}$ since $g_{n,1}$ is not bounded as $n \to \infty$. However, we will implement a method similar to the one in Section 3.2 and

Section 3.3.

Proposition 3.10. *For* $\mathbf{I}_1$ *defined in* (2.7) *and* $z \notin \partial D$ *we have*

$$\mathbf{I}_1 = \frac{\left|\mathrm{tr}_n(A_n - z)G^2(x^2_{\varepsilon,n})\right|^2}{\mathrm{tr}_n G^2(x^2_{\varepsilon,n}) + \varepsilon/(2x^3_{\varepsilon,n})} + O(n^{-\alpha}). \tag{3.26}$$

Proof. Notice that $\partial_{z_1}\mathcal{F}_n(z,z) = \mathrm{tr}_n\big((\bar{z} - A_n^*)G(u_1^2 + u_2^2)\big)$. Denote $p_1(x) = \mathrm{tr}_n\big((\bar{z} - A_n^*)G(x)\big)$, $p_2(x) = \mathrm{tr}_n\big((z - A_n)G(x)\big)$. We have:

$$\begin{aligned}\mathbf{I}_1 = p_1(x^2_{\varepsilon,n})\Big\langle \Big(\partial_{\bar{z}} + \partial_{\bar{z}_1}\Big)\Big(\varphi_n(z,z_1)\, e^{n\mathcal{F}_n(z,z_1)}\Big)\Big\rangle\Big|_{z_1=z} +\\ + \Big\langle \big(p_1(u_1^2+u_2^2) - p_1(x^2_{\varepsilon,n})\big)\Big(\partial_{\bar{z}} + \partial_{\bar{z}_1}\Big)\Big(\varphi_n(z,z_1)\, e^{n\mathcal{F}_n(z,z_1)}\Big)\Big\rangle\Big|_{z_1=z}\end{aligned} \tag{3.27}$$

Recall that $\mathcal{Z}(\varepsilon,\varepsilon,z,z) = 1$, hence $(\partial_{\bar{z}} + \partial_{\bar{z}_1})\,\mathcal{Z}(\varepsilon,\varepsilon,z,z_1)\Big|_{z=z_1} = 0$. The last identity can be rewritten as

$$\Big\langle \Big(\partial_{\bar{z}} + \partial_{\bar{z}_1}\Big)\Big(\varphi_n(z,z_1)\, e^{n\mathcal{F}_n(z,z_1)}\Big)\Big\rangle\Big|_{z_1=z} = 0. \tag{3.28}$$

Combining (3.27) and (3.28), we get

$$\mathbf{I}_1 = \Big\langle \big(p_1(u_1^2+u_2^2) - p_1(x^2_{\varepsilon,n})\big)\Big(\partial_{\bar{z}} + \partial_{\bar{z}_1}\Big)\Big(\varphi_n(z,z_1)\, e^{n\mathcal{F}_n(z,z_1)}\Big)\Big\rangle\Big|_{z_1=z}$$

One can check that

$$\begin{aligned}\mathbf{I}_1 = n\big\langle\big(p_1(u_1^2+u_2^2) - p_1(x^2_{\varepsilon,n})\big)\,\big(p_2(u_1^2+u_2^2) - p_2(s^2-t^2) + O(n^{-1})\big)\times\\ \times\,\varphi_n(z,z)e^{n\mathcal{F}_n(z,z)}\big\rangle\end{aligned}$$

First consider the case $z \in \mathrm{Int}\, D$. We can move contours, restrict the integration and change the variables as in Section 3.2. After the change of variables we get

$$\begin{aligned}\mathbf{I}_1 =&n\int d\widetilde{u}\, d\widetilde{t}\, d\widetilde{s}\, d\widetilde{r}\, d\widetilde{v}\, d\widetilde{w}\, \big(p_1(u^2)-p_1(x^2_{\varepsilon,n})\big)\times\\ &\times\Big(\big(p_2(u^2)-p_2(x^2_{\varepsilon,n})\big)-\big(p_2(s^2-t^2)-p_2(x^2_{\varepsilon,n})\big)\Big)\Phi_n(u,s,t,r,v,w)\times\\ &\times\exp\{-\kappa_1\widetilde{u}^2-\kappa_3\widetilde{s}^2-\kappa_1\widetilde{t}^2-\widetilde{r}^2-\varepsilon\widetilde{v}^2-2\varepsilon\widetilde{w}^2+O(n^{-1/2}\log^k n)\}\end{aligned}$$

Note that $p_1(u^2)-p_1(x^2_{\varepsilon,n})$, $p_2(u^2)-p_2(x^2_{\varepsilon,n})$ and $p_2(s^2-t^2)-p_2(x^2_{\varepsilon,n})$ have zeros of the first order at the saddle point, thus the multiplier n before the integral will vanish. Taking into account that

$$\begin{aligned}&p_j(u^2)-p_j(x^2_{\varepsilon,n})=n^{-1/2}\gamma_j\widetilde{u}+O(n^{-1}\log^k n);\\ &p_2(s^2-t^2)-p_2(x^2_{\varepsilon,n})=-n^{-1/2}i\gamma_2\widetilde{t}+O(n^{-1}\log^k n),\end{aligned}$$

where $\gamma_1=-2x_{\varepsilon,n}\cdot\mathrm{tr}_n(\bar{z}-A_n^*)G^2(x^2_{\varepsilon,n})$, $\gamma_2=-2x_{\varepsilon,n}\cdot\mathrm{tr}_n(z-A_n)G^2(x^2_{\varepsilon,n})$, we obtain

$$\begin{aligned}\mathbf{I}_1 =&\Phi_n(x_{\varepsilon,n},0,ix_{\varepsilon,n},0,0,0)\cdot\int d\widetilde{u}\, d\widetilde{t}\, d\widetilde{s}\, d\widetilde{r}\, d\widetilde{v}\, d\widetilde{w}\, (\gamma_1\gamma_2\cdot\widetilde{u}^2+i\gamma_1\gamma_2\cdot\widetilde{u}\widetilde{t})\times\\ &\times\exp\{-\kappa_1\widetilde{u}^2-\kappa_3\widetilde{s}^2-\kappa_1\widetilde{t}^2-\widetilde{r}^2-\varepsilon\widetilde{v}^2-2\varepsilon\widetilde{w}^2\}+O(n^{-1/2}\log^k n).\end{aligned}\tag{3.29}$$

Since the gaussian integral $\int x^n e^{-kx^2}\,dx$ equals to zero for odd n, we can omit the summand $i\gamma_1\gamma_2\cdot\widetilde{u}\widetilde{t}$. Now recall that (2.6) holds and we can write

$$\begin{aligned}1=&\Big\langle\,\varphi_n(z,z)e^{n\mathcal{F}_n(z,z)}\Big\rangle=\Phi_n(x_{\varepsilon,n},0,ix_{\varepsilon,n},0,0,0)\times\\ &\times\int d\widetilde{u}\, d\widetilde{t}\, d\widetilde{s}\, d\widetilde{r}\, d\widetilde{v}\, d\widetilde{w}\,\exp\{-\kappa_1\widetilde{u}^2-\kappa_3\widetilde{s}^2-\kappa_1\widetilde{t}^2-\widetilde{r}^2-\varepsilon\widetilde{v}^2-2\varepsilon\widetilde{w}^2\}+\\ &+O(n^{-1/2}\log^k n).\end{aligned}\tag{3.30}$$

Dividing (3.29) by (3.30), we obtain

$$\mathbf{I}_1 = \frac{\gamma_1\gamma_2 \int \widetilde{u}^2 \exp\{-\kappa_1 \widetilde{u}^2\}}{\int \exp\{-\kappa_1 \widetilde{u}^2\}} + O(n^{-1/3}) = \frac{\gamma_1\gamma_2}{2\kappa_1} + O(n^{-1/3}) =$$
$$= \frac{\mathrm{tr}_n (A_n - z)^* G^2(x^2_{\varepsilon,n}) \cdot \mathrm{tr}_n (A_n - z) G^2(x^2_{\varepsilon,n})}{\mathrm{tr}_n G^2(x^2_{\varepsilon,n}) + \varepsilon/(2x^3_{\varepsilon,n})} + O(n^{-1/3}).$$

Taking into account $\mathrm{tr}_n (A_n - z)^* G^2(x^2_{\varepsilon,n}) = \overline{\mathrm{tr}_n (A_n - z) G^2(x^2_{\varepsilon,n})}$, we obtain the formula for $\mathbf{I}_1$.

In the case $z \in \mathbb{C} \setminus \overline{D}$, we can move contours, restrict the integration and change the variables as in Section 3.3. Making similar computations as above and using the fact that $\displaystyle\int \widetilde{u}_1 \exp\{-\kappa_1 \widetilde{u}_1^2\}\, d\widetilde{u}_1 = 0$, we will obtain the same result.

□

Substituting (3.25) and (3.26) into the formula from Proposition 2.2, we obtain

$$\pi \cdot \rho_{\varepsilon,n,\omega}(z) = \frac{\left|\mathrm{tr}_n (A_n - z) G^2(x^2_{\varepsilon,n})\right|^2}{\mathrm{tr}_n G^2(x^2_{\varepsilon,n}) + \varepsilon/(2x^3_{\varepsilon,n})} + x^2_{\varepsilon,n} \cdot \mathrm{tr}_n G(x^2_{\varepsilon,n}) \widetilde{G}(x^2_{\varepsilon,n}) + \frac{\beta(\varepsilon,n)}{n^\alpha}, \tag{3.31}$$

where $|\beta(\varepsilon,n)| \leq C(\varepsilon)$ as $n \to \infty$.

Chapter 4

Final formula for $\rho_\omega(z)$

Using asymptotic formula for $\rho_{\varepsilon,n,\omega}(z)$, we complete the proof of Theorem 0.1 by following the plan described in Remark 1.1.

4.1. Case $z \in \operatorname{Int} D$

We are going to take the limit in (3.31) as $n \to \infty$ for fixed $\varepsilon \in (0, \varepsilon_0)$ and $z \in \operatorname{Int} D$. First we show that

$$\lim_{n\to\infty} \operatorname{tr}_n (A_n - z) G^2(x^2_{\varepsilon,n}) = T_1(z, x^2_\varepsilon). \tag{4.1}$$

Denote $f_n(x) = \operatorname{tr}_n(A_n - z)G^2(x^2)$, $f(x) = T_1(z, x^2)$. Observe that

$$|f_n(x_{\varepsilon,n}) - f(x_\varepsilon)| \le |f'_n(\xi_n)| \cdot |x_{\varepsilon,n} - x_\varepsilon| + |f_n(x_\varepsilon) - f(x_\varepsilon)|. \tag{4.2}$$

One can show that $|f'_n(x)|$ is bounded. Now Proposition 3.2 and (C5) imply that $f_n(x_{\varepsilon,n}) \to f(x_\varepsilon)$ as $n \to \infty$. Similarly one can prove the relations

$$\begin{aligned}
&\lim_{n\to\infty} \operatorname{tr}_n G(x^2_{\varepsilon,n}) \widetilde{G}(x^2_{\varepsilon,n}) = T_2(z, x^2_\varepsilon), \\
&\lim_{n\to\infty} \operatorname{tr}_n G^2(x^2_{\varepsilon,n}) = \int (\lambda + x^2_\varepsilon)^{-2} \, d\nu_z(\lambda).
\end{aligned}$$

Thus, for $\rho_{\varepsilon,\omega}(z) = \lim\limits_{n\to\infty} \rho_{\varepsilon,n,\omega}(z)$ we obtain the following expression:

$$\pi \cdot \rho_{\varepsilon,\omega}(z) = \frac{|T_1(z, x^2_\varepsilon)|^2}{\int (\lambda + x^2_\varepsilon)^{-2} \, d\nu_z(\lambda) + \varepsilon/(2x^3_\varepsilon)} + x^2_\varepsilon \cdot T_2(z, x^2_\varepsilon). \tag{4.3}$$

We are left to find the limit $\rho_\omega(z) = \lim_{\varepsilon\to 0} \rho_{\varepsilon,\omega}(z)$. According to Proposition 3.2, $\lim_{\varepsilon\to 0} x_\varepsilon = x_0$. Clearly, the function $\int (\lambda + x^2)^{-2}\, d\nu_z(\lambda)$ is continuous for $x > 0$. Now consider $T_1(z, x^2)$. Since the derivative of $f_n(x) = \mathrm{tr}_n(A_n - z)G^2(x^2)$ is bounded uniformly in $x \geq \kappa_0$, which shows that $f_n(x)$ are equicontinuous uniformly for $x \geq \kappa_0$. Recall that a pointwise convergent sequence of uniformly equicontinuous functions on a metric space converges to a continuous function (it is an immediate consequence of Arzelà-Ascoli theorem). Applying this observation to the sequence $f_n(x)$, we get that $T_1(z, x^2)$ is continuous in x for $x \geq \kappa_0$. Similarly, $T_2(z, x^2)$ is continuous in x for $x \geq \kappa_0$. Thus

$$\pi \cdot \rho_\omega(z) = \frac{|T_1(z, x_0^2)|^2}{\int (\lambda + x_0^2)^{-2}\, d\nu_z(\lambda)} + x_0^2 \cdot T_2(z, x_0^2), \tag{4.4}$$

which shows that $\rho_\omega(z)$ is equal to $\rho(z)$ defined in (0.4) for $z \in \mathrm{Int}\, D$.

Also we need to check that $\rho_{\varepsilon,\omega}(z)$ is bounded uniformly in $\varepsilon \leq \varepsilon_0$ and $z \in \mathrm{Int}\, D$. To estimate $T_j(z, x_{\varepsilon,n}^2)$, it is enough to obtain upper bounds on $\mathrm{tr}_n(A_n - z)G^2(x_{\varepsilon,n}^2)$ and $\mathrm{tr}_n G(x_{\varepsilon,n}^2)\widetilde{G}(x_{\varepsilon,n}^2)$. Observe that $\mathrm{dist}(\sigma_\epsilon, \partial D) > 0$, where σ_ϵ is defined in (C3). Now one can easily obtain upper bounds on $\mathrm{tr}_n(A_n - z)G^2(x_{\varepsilon,n}^2)$, $\mathrm{tr}_n G(x_{\varepsilon,n}^2)\widetilde{G}(x_{\varepsilon,n}^2)$ and a lower bound on $\int (\lambda + x_\varepsilon^2)^{-2}\, d\nu_z(\lambda)$, using (C2), Proposition 3.2 for $z \in \sigma_\epsilon$ (taking $E_{in} = \sigma_\epsilon$) and (C3) for $z \notin \sigma_\epsilon$.

4.2. Case $z \in \mathbb{C} \setminus \overline{D}$

Similarly to the case $z \in \mathrm{Int}\, D$ we obtain

$$\pi \cdot \rho_{\varepsilon,\omega}(z) = \frac{|T_1(z, x_\varepsilon^2)|^2}{\int (\lambda + x_\varepsilon^2)^{-2}\, d\nu_z(\lambda) + \varepsilon/(2x_\varepsilon^3)} + x_\varepsilon^2 \cdot T_2(z, x_\varepsilon^2).$$

To find the limit $\rho_\omega(z) = \lim\limits_{\varepsilon\to 0} \rho_{\varepsilon,\omega}(z)$, use the bound

$$\pi \cdot \rho_{\varepsilon,\omega}(z) \le \frac{|T_1(z,x_\varepsilon^2)|^2}{\varepsilon/(2x_\varepsilon^3)} + x_\varepsilon^2 \cdot T_2(z,x_\varepsilon^2) = |T_1(z,x_\varepsilon^2)|^2 \cdot 2x_\varepsilon^2 \cdot \frac{x_\varepsilon}{\varepsilon} + x_\varepsilon^2 \cdot T_2(z,x_\varepsilon^2). \tag{4.5}$$

According to Proposition 3.4, $x_\varepsilon \le (1+K_0)\varepsilon$ for a fixed $z \in \mathbb{C}\setminus\overline{D}$. Using condition (C3) one can show that $|T_{1,2}(z,x_\varepsilon^2)|$ are bounded uniformly in ε. Then (4.5) gives us

$$|\rho_{\varepsilon,\omega}(z)| \le C\varepsilon^2 \tag{4.6}$$

for some $C>0$, which shows that $\rho_\omega(z) = \lim\limits_{\varepsilon\to 0} \rho_{\varepsilon,\omega}(z) = 0$, and $\rho_\omega(z)$ is equal to $\rho(z)$ defined in (0.4) for $z \in \mathbb{C}\setminus D$.

Again, we need to prove that $\rho_{\varepsilon,\omega}(z)$ is bounded uniformly in $\varepsilon \le \varepsilon_0$ and $z \in \mathbb{C}\setminus\overline{D}$: $|z| \le C$. Proposition 3.4 and Proposition 3.5 imply that $x_\varepsilon \le C\varepsilon^{1/3}$ for $z \notin D$, $|z| \le C$. We obtain a bound

$$\pi \cdot \rho_{\varepsilon,\omega}(z) \le \frac{|T_1(z,x_\varepsilon^2)|^2}{\varepsilon/(2x_\varepsilon^3)} + x_\varepsilon^2 \cdot T_2(z,x_\varepsilon^2) \le C|T_1(z,x_\varepsilon^2)|^2 + C\varepsilon^{2/3} T_2(z,x_\varepsilon^2). \tag{4.7}$$

The fact that $|T_j(z,x_\varepsilon^2)|$ are uniformly bounded follows from (C3) and (C2).

Chapter 5

Proof of Proposition 1.1

Recall that we have a random matrix $X_n = A_n + H_n$ with eigenvalues $z_1, \dots, z_n$ and $Y(z) = (X_n - z)(X_n - z)^*$. Also, we have a smooth function $h(z)$ on $\mathbb{C}$ with a compact support E. In Chapter 1 we noted that $\displaystyle\int h(z) \cdot \frac{1}{4\pi n} \Delta \log\det Y(z)\, \mathbf{d}^2 z = \frac{1}{n}\sum_{j=1}^{n} h(z_j)$. Thus it suffices to prove that

$$\lim_{\varepsilon\to 0} \int_E h(z) \cdot \frac{1}{4\pi n} \Delta\, \mathbb{E}_{H_n}\Big\{\log\det(Y(z)+\varepsilon^2)\Big\}\, \mathbf{d}^2 z = \\ = \int_E h(z) \cdot \frac{1}{4\pi n} \Delta\, \mathbb{E}_{H_n}\Big\{\log\det Y(z)\Big\}\, \mathbf{d}^2 z$$

uniformly in n for $n \geq n_0$, which is equivalent to the fact that

$$\lim_{\varepsilon\to 0} \frac{1}{4\pi n}\, \mathbb{E}_{H_n}\Big\{\int_E \Delta h(z) \cdot \log\det(Y(z)+\varepsilon^2)\, \mathbf{d}^2 z\Big\} = \\ = \frac{1}{4\pi n}\, \mathbb{E}_{H_n}\Big\{\int_E \Delta h(z) \cdot \log\det Y(z)\, \mathbf{d}^2 z\Big\} \tag{5.1}$$

uniformly in n for $n \geq n_0$.

One can check that for a fixed n we have $\lim\limits_{\varepsilon\to 0} \log\det(Y(z)+\varepsilon^2) = \log\det Y(z)$ and estimate $\log\det(Y(z)+\varepsilon^2)$ by some integrable over E function with finite H_n-expectation. Then dominated convergence theorem implies that (5.1) holds for each n. We are left to prove that convergence (5.1) is uniform in n.

It suffices to prove that $\Phi(\varepsilon, n, z) = \dfrac{1}{n}\, \mathbb{E}_{H_n}\Big\{\log\det(Y(z)+\varepsilon^2)\Big\}$ converges uniformly for $n \geq n_0$ and $z \in E$ as $\varepsilon \to 0$. Set

$$T(\varepsilon, n, z) := \partial_\varepsilon \Phi(\varepsilon, n, z) = \mathbb{E}_{H_n}\Big\{2\varepsilon \cdot \mathrm{tr}_n (Y(z) + \varepsilon^2)^{-1}\Big\}.$$

Similarly to (1.4) one can prove that $T(\varepsilon, n, z) = \dfrac{1}{n}\Big(\partial_{\varepsilon_1} \mathcal{Z}(\varepsilon, \varepsilon_1, z, z)\Big)\Big|_{\varepsilon_1 = \varepsilon}$, where $\mathcal{Z}(\varepsilon, \varepsilon_1, z, z)$ is defined in (1.5). By differentiating (2.1) with respect to ε_1, we obtain the following integral representation:

$$\begin{aligned} T(\varepsilon, n, z) = &-\frac{4n^3}{\pi^3} \int_0^\infty dR \int_{-\infty}^\infty dv\, du_1\, du_2\, ds \int_L dt \cdot \frac{R}{\sqrt{v^2 + 4R}} \times \\ &\times \varphi(u_1^2 + u_2, s^2 - t^2, z, z) \cdot (u_1 + \varepsilon) \times \\ &\times \exp\Big\{n\Big(\mathcal{L}_n(u_1^2 + u_2^2) - (u_1 + \varepsilon)^2 - u_2^2\Big)\Big\} \times \\ &\times \exp\Big\{-n\Big(\mathcal{L}_n(s^2 - t^2) + (t - i\varepsilon)^2 + (R + it + \varepsilon)^2 + \varepsilon v^2\Big)\Big\}, \end{aligned} \tag{5.2}$$

Suppose that $T(\varepsilon, n, z) \leq \dfrac{C}{\varepsilon^{1-\alpha}}$ for $\varepsilon \leq \varepsilon_0$, $n \geq n_0$, $z \in E$ and some fixed $\alpha, C > 0$. Then, for $\varepsilon_1 < \varepsilon_2 \leq \varepsilon_0$ we have

$$|\Phi(\varepsilon_2, n, z) - \Phi(\varepsilon_1, n, z)| = \Big|\int_{\varepsilon_1}^{\varepsilon_2} T(\varepsilon, n, z)\, d\varepsilon\Big| \leq C \int_{\varepsilon_1}^{\varepsilon_2} \frac{d\varepsilon}{\varepsilon^{1-\alpha}} \leq \frac{C}{\alpha} \varepsilon_2^\alpha,$$

which means that $\Phi(\varepsilon, n, z)$ converges uniformly for $n \geq n_0$, $z \in E$ as $\varepsilon \to 0$.

It suffices to prove the following facts:

Theorem 5.1. *Fix an arbitrary $d > 0$ and set $E_d := \{z \in E \mid \mathrm{dist}(z, \partial D) \geq d\}$. Then there exist n_0, $\varepsilon_0 > 0$ and $C > 0$ such that $\varepsilon^{1/2}\, |T(\varepsilon, n, z)| \leq C$ for $n \geq n_0$, $0 < \varepsilon \leq \varepsilon_0$ and $z \in E_d$.*

Theorem 5.2. *There exist $d > 0$, n_0, $\varepsilon_0 > 0$ and $C > 0$ such that $\varepsilon^{3/4}\, |T(\varepsilon, n, z)| \leq C$ for $n \geq n_0$, $0 < \varepsilon \leq \varepsilon_0$ and $z \in E \setminus E_d$.*

We split the proofs of the theorems above into several parts depending on the size of ε with respect to n and the location of z.

5.1. Case $\varepsilon < a/n$

We start with the following result:

Proposition 5.3. *Fix an arbitrary* $a > 0$, $d > 0$, *and set* $E_{d,in} := \{z \in E \mid z \in D,\ \mathrm{dist}(z, \partial D) \geq d\}$. *Then there exist* $n_0 \in \mathbb{N}$ *and* $C > 0$ *such that* $\varepsilon^{1/2}\,|T(\varepsilon, n, z)| \leq C$ *for* $n \geq n_0$, $0 < \varepsilon < \dfrac{a}{n}$ *and* $z \in E_{d,in}$.

Proof. Since $E_{d,in}$ is a compact subset of $\mathrm{Int}\, D$, we can use the results of Section 3.1 for $E_{in} = E_{d,in}$.

Set $\widehat{\varepsilon} := n\varepsilon$, then $0 < \widehat{\varepsilon} < a$. Make a change of variables $(u_1, u_2) \to (u, \theta)$ where $u_1 = u\cos\theta$, $u_2 = u\sin\theta$, $u \in [0, \infty)$, $\theta \in [0, 2\pi]$. The jacobian of this change $J = u$. Also we can make a change $r = R + it \in \mathcal{R}(t)$, where $\mathcal{R}(t) = \{-it + \tau,\ \tau > 0\}$ and integrate with respect to θ, then

$$\varepsilon^{1/2}\, T(\varepsilon, n, z) = \frac{8n^{5/2}}{\pi^2} \int_0^\infty du \int_{-\infty}^{\infty} dv\, ds \int_L dt \int_{\mathcal{R}(t)} dr \cdot \frac{r - it}{\sqrt{v^2 + 4(r - it)}} \times$$
$$\times\, \varphi_n(u^2, s^2 - t^2, z, z) \cdot \left(u^2 I_1(2\widehat{\varepsilon}u) - \frac{\widehat{\varepsilon}}{n} u I_0(2\widehat{\varepsilon}u)\right) \cdot \sqrt{\widehat{\varepsilon}}\, e^{-\widehat{\varepsilon}v^2} \times$$
$$\times \exp\left\{-2\widehat{\varepsilon}(r - it) - \widehat{\varepsilon}^2/n\right\} \exp\left\{n\Big(\mathcal{L}_n(u^2) - u^2 - \mathcal{L}_n(s^2 - t^2) - t^2 - r^2\Big)\right\},$$

where $I_0(x) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} e^{x\cos\theta} d\theta$, $I_1(x) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \cos(\theta) e^{x\cos\theta} d\theta$ are modified Bessel functions.

Now we change the t-contour and r-contour to L_t and $\widetilde{\mathcal{R}}(t)$ respectively, which are defined further in (5.7). We will see that $\Re(r - it) \geq 0$ for $r \in \widetilde{\mathcal{R}}(t)$, thus $\left|\dfrac{r - it}{\sqrt{v^2 + 4(r - it)}}\right| \leq \dfrac{|\sqrt{r - it}|}{2}$. After integrating with respect to v we obtain

$$\varepsilon^{1/2}\,|T(\varepsilon, n, z)| \leq C n^{5/2} \int_{\mathcal{V}} \Phi_n(\mathbf{x})\, e^{n\Re F_n(\mathbf{x})}\, d\mathbf{x}, \qquad (5.3)$$

where

$$\begin{aligned}
&\mathbf{x} = (u,s,t,r), \quad \mathcal{V} = [0,+\infty) \times \mathbb{R} \times L_t \times \widetilde{\mathcal{R}}(t), \\
&F_{n,1}(u) = \mathcal{L}_n(u^2) - u^2, \quad F_{n,2}(s,t,r) = -\mathcal{L}_n(s^2-t^2) - t^2 - r^2, \\
&F_n(\mathbf{x}) = F_{n,1}(u) + F_{n,2}(s,t,r), \quad \widetilde{\varphi}(u,s,t) = \varphi_n(u^2, s^2-t^2, z, z), \\
&\Phi_n(u,s,t,r) = \sqrt{|r-it|} \cdot |\widetilde{\varphi}_n(u,t,s)| \cdot (u^2 I_1(2au) + \frac{a}{n} u I_0(2au)).
\end{aligned} \tag{5.4}$$

We can study $F_{n,1}$ and $F_{n,2}$ as in Section 3.2.

Notice that $F'_{n,1}(u) = 2u\,(\mathrm{tr}_n G(u^2) - 1)$ has exactly one positive root $x_{0,n}$, thus $u = x_{0,n}$ is the maximum point of $F_{n,1}(u)$. For u lying in some neighbourhood of $x_{0,n}$ we have

$$F_{n,1}(u) = \mathcal{L}_n(x_{0,n}^2) - x_{0,n}^2 - K_1(u - x_{0,n})^2 + O(|u - x_{0,n}|^3),$$

where

$$K_1 = 2x_{0,n}^2 \cdot \mathrm{tr}_n G^2(x_{0,n}^2). \tag{5.5}$$

Proposition 3.2 shows that $K_1 \geq C > 0$ uniformly in ε, n and $z \in E_{d,in}$, thus

$$F_{n,1}(u) \leq F_{n,1}(x_{0,n}) - Cn^{-1}\log^2 n \quad \text{when } |u - x_{0,n}| > n^{-1/2}\log n \tag{5.6}$$

For $F_{n,2}(s,t,r)$ we can prove the analogue of Lemma 3.6: there exist constants δ, σ, C_0 and a contour L_1 which connects points $ix_{0,n}+\delta$ and C_0+iC_0 satisfying the following conditions:

- $0 < \delta < \dfrac{\kappa_0}{2}$, $\sigma > 0$, $C_0 > 0$ and $\log 2C_0^2 > \mathcal{L}_n(x_{0,n}^2) + 1$;
- δ, σ, C_0 do not depend on n and $z \in E$;
- $L_1 \subset \{w \mid \Re w > 0, \Im w \geq \Re w\}$;
- $\Re h_n(ix_{0,n} + t)$ decreases for $t \in [0, \delta]$, and $\Re h_n(t) \leq \Re h_n(ix_{0,n}) - \sigma$ for $t \in L_1$.

Now take L_2 to be symmetric to L_1 with respect to the imaginary axis, and change t-contour and r-contour to the following contours:

$$\begin{aligned} L_t =& L_0 \cup L_1 \cup L_2 \cup L_1' \cup L_2', \text{ where } L_0 = [ix_{0,n} - \delta, ix_{0,n} + \delta], \\ & L_{1,2}' = \{(iC_0 \pm C_0 \pm \tau, \tau > 0\}; \\ \widetilde{\mathcal{R}}(t) =& \mathcal{R}_1(t) \cup \mathcal{R}_2(t), \quad \text{where} \quad \mathcal{R}_1(t) = [it, -\delta], \\ & \mathcal{R}_2(t) = \{r \colon r = -\delta + \rho, \rho \geq 0\}. \end{aligned} \tag{5.7}$$

As in Section 3.2 we can show that

$$\begin{aligned} \Re F_{n,2}(s,t,r) \leq \Re F_{n,2}(0, ix_{0,n}, 0) - Cn^{-1}\log^2 n \quad \text{when} \\ \max\{|s|, |t - ix_{0,n}|, |r|\} > n^{-1/2}\log n, \end{aligned} \tag{5.8}$$

and we can also expand $F_{n,2}(s,t,r)$ in some neighbourhood of $(0, ix_{0,n}, 0)$:

$$F_{n,2}(s,t,r) = -\mathcal{L}_n(x_{0,n}^2) + x_{0,n}^2 - K_1(t - ix_{0,n})^2 - s^2 - r^2 + O(|t - ix_{0,n}|^3 + |s|^3),$$

where K_1 is defined in (5.5). The estimations (5.6) and (5.8) together with the fact that (5.3) does not depend on $\widehat{\varepsilon}$ allows us to restrict the integration to a neighbourhood of $(x_{0,n}, 0, ix_{0,n}, 0)$. After the following change of variables: $u = x_{0,n} + \widetilde{u}n^{-1/2}, \quad s = \widetilde{s}n^{-1/2}, \quad t = ix_{0,n} + \widetilde{t}n^{-1/2}, \quad r = \widetilde{r}n^{-1/2}$, we obtain

$$\begin{aligned} \varepsilon^{1/2}\,|T(\varepsilon, n, z)| \leq \frac{4n^{1/2}}{\pi^{3/2}} \int_{U_n} \Phi_n(x_{0,n} + \widetilde{u}n^{-1/2}, \widetilde{s}n^{-1/2}, ix_{0,n} + \widetilde{t}n^{-1/2}, \widetilde{r}n^{-1/2}) \times \\ \times \exp\{-K_1\widetilde{u}^2 - K_1\widetilde{t}^2 - \widetilde{s}^2 - \widetilde{r}^2 + O(n^{-1/2}\log^3 n)\} + O(e^{-C\log^2 n}), \end{aligned}$$

where $U_n = \{(\widetilde{u}, \widetilde{s}, \widetilde{t}, \widetilde{r}) \in \mathbb{R}^4 \colon |\widetilde{u}|,\ |\widetilde{s}|,\ |\widetilde{t}|,\ |\widetilde{r}| < \log n\}$. The multiplier $n^{1/2}$ before the integral will vanish since Φ_n has a root at the saddle point. More precisely, the identity in Remark 2.1 implies that $\widetilde{\varphi}_n(x_{0,n}, 0, ix_{0,n}) = O(n^{-1})$, thus $|\widetilde{\varphi}_n(x_{0,n} + \widetilde{u}n^{-1/2}, \widetilde{s}n^{-1/2}, ix_{0,n} + \widetilde{t}n^{-1/2})| \leq |\phi_u||\widetilde{u}|n^{-1/2} + |\phi_t||i\widetilde{t}|n^{-1/2} + O(n^{-1}\log^2 n)$, where $\phi_u = \partial_u \widetilde{\varphi}_n(x_{0,n}, 0, ix_{0,n})$, $\phi_t = \partial_t \widetilde{\varphi}_n(x_{0,n}, 0, ix_{0,n})$, and the other multipliers in Φ_n are bounded in the neighbourhood of $(x_{0,n}, 0, ix_{0,n}, 0)$. We obtain:

$$\varepsilon^{1/2} |T(\varepsilon, n, z)| \leq C \int_{\widetilde{U}_n} (|\phi_u||\widetilde{u}| + |\phi_t||\widetilde{t}|) \cdot e^{-K_1\widetilde{u}^2 - K_1\widetilde{t}^2 - \widetilde{s}^2 - \widetilde{r}^2} \, d\widetilde{u} \, d\widetilde{t} \, d\widetilde{s} \, d\widetilde{r}$$

and it is easy to check that the integral above is bounded.

$\square$

A similar result holds for $z \in E$ lying outside of D far enough from ∂D. More precisely,

Proposition 5.4. *Fix an arbitrary* $a > 0$, $d > 0$, *and set* $E_{d,out} := \{z \in E \mid z \notin D, \ \mathrm{dist}(z, \partial D) \geq d\}$. *Then there exist* $n_0 \in \mathbb{N}$ *and* $C > 0$ *such that* $\varepsilon^{1/2} |T(\varepsilon, n, z)| \leq C$ *for* $n \geq n_0$, $0 < \varepsilon < \dfrac{a}{n}$ *and* $z \in E_{d,out}$.

Proof. Since $E_{d,out}$ is a compact subset of $\mathbb{C} \setminus \overline{D}$, we can use the results of Section 3.1 for $E_{out} = E_{d,out}$.

Similarly to the proof of Proposition 5.3 we obtain that

$$\varepsilon^{1/2} |T(\varepsilon, n, z)| \leq C n^{5/2} \int_{\mathcal{V}} \Phi_n(\mathbf{x}) \, e^{n \Re F_n(\mathbf{x})} \, d\mathbf{x},$$

with the same notations as in (5.4). We also change the t-contour and r-contour as follows:

$$\begin{aligned} &\mathbf{L}_t = \mathbf{L}_1 \cup \mathbf{L}_2 \cup \mathbf{L}_1' \cup \mathbf{L}_2', \quad \text{where} \quad \mathbf{L}_{1,2} = \{((\pm 1 + i)\tau, 0 \leq \tau < C_1\}, \\ &\qquad \mathbf{L}_{1,2}' = \{(\pm 1 + i)C_1 \pm \tau, \tau > 0\}; \\ &\widetilde{\mathcal{R}}(t) = \mathcal{R}_1(t) \cup \mathcal{R}_2(t), \quad \text{where} \quad \mathcal{R}_1(t) = [it, -\delta], \\ &\qquad \mathcal{R}_2(t) = \{r \colon r = -\delta + \rho, \rho \geq 0\} \end{aligned} \tag{5.9}$$

The contour $\mathbf{L}_t$ here is built similar to t-contour in Section 3.3, but for $\varepsilon =$

0. One can check that for small u, s, t, r we have

$$F_{n,1}(u) = \mathcal{L}_n(0) - K_2u^2 + O(u^4);$$
$$F_{n,2}(s,t,r) = -\mathcal{L}_n(0) - K_3s^2 - iK_2|t|^2 - K_4|t|^4 - r^2 + O(|s|^3 + |t|^6);$$

where

$$K_2 = 1 - \mathrm{tr}_n G(0), \quad K_3 = \mathrm{tr}_n G(0), \quad K_4 = \frac{1}{2}\mathrm{tr}_n G^2(0). \tag{5.10}$$

Obviously, $K_2, K_3, K_4 \geq C > 0$ uniformly in n, hence

$$\begin{aligned} F_{n,1}(u) &\leq \mathcal{L}_n(0) - Cn^{-1}\log^2 n, \quad \text{when } |u| > n^{-1/2}\log n; \\ \Re F_{n,2}(s,t,r) &\leq -\mathcal{L}_n(0) - Cn^{-1}\log^2 n, \text{ when } \max\{|s|,|r|\} > n^{-1/2}\log n \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{or } |t| > n^{-1/4}\log^{1/2} n, \end{aligned}$$

which shows that we can restrict the integration to $|u|$, $|s|$, $|r| < n^{-1/2}\log n$, $|t| < n^{-1/4}\log^{1/2} n$. After a change of variables $u = n^{-1/2}\widetilde{u}$, $s = n^{-1/2}\widetilde{s}$, $r = n^{-1/2}\widetilde{r}$, $t = n^{-1/4}(\pm 1 + i)\widetilde{t}_\pm$ we obtain:

$$\begin{aligned} \varepsilon^{1/2}\,|T(\varepsilon,n,z)| \leq Cn^{3/4} &\int d\widetilde{u}\, d\widetilde{s}\, d\widetilde{t}_\pm\, d\widetilde{r}\ \Phi_n(\widetilde{\mathbf{x}})\times \\ &\times \exp\{-K_2\widetilde{u}^2 - K_3\widetilde{s}^2 - 4K_4\widetilde{t}_\pm^4 - \widetilde{r}^2 + O(n^{-1/2}\log^k n)\} \ + O(e^{-C\log^2 n}), \end{aligned}$$

It is easy to see that $\Phi_n(n^{-1/2}\widetilde{u}, n^{-1/2}\widetilde{s}, n^{-1/4}(1+i)\widetilde{\tau}, n^{-1/2}\widetilde{r}) \leq Cn^{-1}$ since it has a multiplier $u^2 I_1(2au) + \dfrac{a}{n}uI_0(2au)$, thus the integral above is bounded. $\square$

Next, we study the case when z is close to the boundary ∂D.

Proposition 5.5. *Fix an arbitrary $a > 0$. Then there exist $d > 0$, $n_0 \in \mathbb{N}$ and $C > 0$ such that $\varepsilon^{1/2}\,|T(\varepsilon,n,z)| \leq C$ for $n \geq n_0$, $0 < \varepsilon < \dfrac{a}{n}$ and $z \in E_{d,\partial}$, where $E_{d,\partial} = \{z \in E \mid \mathrm{dist}(z,\partial D) \leq d\}$.*

Proof. Again, we have $\varepsilon^{1/2}\,|T(\varepsilon,n,z)| \leq Cn^{5/2}\displaystyle\int_{\mathcal{V}} \Phi_n(\mathbf{x})\, e^{n\Re F_n(\mathbf{x})}\, d\mathbf{x}$, with the same notations as in (5.4). We split the proof into four cases:

Case 1. $1 - \mathrm{tr}_n G(0) \geq n^{-1/2}$. In this case change t-contour and r-contour as in (5.9), then for small u, t, s, r we have

$$\begin{aligned} &F_{n,1}(u) = \mathcal{L}_n(0) - K_2 u^2 + O(u^4);\\ &F_{n,2}(s,t,r) = -\mathcal{L}_n(0) - K_3 s^2 - iK_2|t|^2 - K_4|t|^4 - r^2 + O(|s|^3 + |t|^6); \end{aligned}$$

where $K_{2,3,4}$ are defined in (5.10). We have $K_2 \geq n^{-1/2}$, $K_3, K_4 \geq C > 0$. Thus the following inequalities hold:

$$\begin{aligned} nF_{n,1}(u) &\leq n\mathcal{L}_n(0) - C\log^2(K_2 n), \quad \text{when } u > (K_2 n)^{-1/2}\log(K_2 n);\\ n\Re F_{n,2}(s,t,r) &\leq -n\mathcal{L}_n(0) - C\log^2 n, \quad \text{when } \max\{|s|,|r|\} > n^{-1/2}\log n \text{ or}\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad |t| > n^{-1/4}\log^{1/2} n. \end{aligned}$$

Since $K_2 n \geq n^{1/2}$, the above bounds allow to restrict the integration to the neighbourhood of the saddle point $(0,0,0,0)$ with the error term of the order $O(e^{-c\log^2 n})$. Make a change $u = (K_2 n)^{-1/2}\widetilde{u}$, $t = n^{-1/4}(\pm 1 + i)\widetilde{t}_{\pm}$, $s = n^{-1/2}\widetilde{s}$, $r = n^{-1/2}\widetilde{r}$, then the multiplier before the integral becomes $\dfrac{Cn^{3/4}}{K_2^{1/2}}$. Estimate the multipliers in $\Phi_n(\mathbf{x})$ as follows: $\sqrt{|r - it|} = O(n^{-1/8}\log^k n)$;

$$\begin{aligned} \widetilde{\varphi}_n(u,s,t) &= \widetilde{\varphi}_n(0,0,0) + O(n^{-1/2}\log^k n) + O((K_2 n)^{-1}\log^k(K_2 n)) =\\ &= K_2^2 + O(n^{-1/2}\log^k n);\\ u^2 I_1(2au) + \frac{a}{n}uI_0(2au) &= O((K_2 n)^{-1}\log^k(K_2 n)) +\\ &+ O(n^{-1}\cdot(K_2 n)^{-1/2}\log^k(K_2 n)) = O((K_2 n)^{-1}\log^k n). \end{aligned}$$

Thus $\Phi_n(\mathbf{x}) = O\big(n^{-1/8}\log^k n \cdot (K_2 n^{-1} + n^{-3/2}K_2^{-1})\big)$. We obtain that $\varepsilon^{1/2}|T(\varepsilon,n,z)| \leq C\dfrac{\log^k n}{n^{1/8}}(K_2^{1/2}n^{-1/4} + K_2^{-3/2}n^{-3/4})$, which is bounded since $n^{-1/2} \leq K_2 \leq C$.

Case 2. $0 \le 1 - \mathrm{tr}_n G(0) \le n^{-1/2}$. In this case change t-contour and r-contour as in (5.9), then for small u, t, s, r we have

$$F_{n,1}(u) = \mathcal{L}_n(0) - K_2 u^2 - K_4 u^4 + O(u^6) \le L_n(0) - K_4 u^4 + O(u^6);$$
$$F_{n,2}(s,t,r) = -\mathcal{L}_n(0) - K_3 s^2 - iK_2 |t|^2 - K_4 |t|^4 - r^2 + O(|s|^3 + |t|^6);$$

Since $K_{3,4} \ge C > 0$, the following inequalities hold:

$$nF_{n,1}(u) \le n\mathcal{L}_n(0) - C\log^2 n, \quad \text{when } u > n^{-1/4}\log^{1/2} n;$$
$$n\Re F_{n,2}(s,t,r) \le -n\mathcal{L}_n(0) - C\log^2 n, \quad \text{when } \max\{|s|,|r|\} > n^{-1/2}\log n \text{ or } |t| > n^{-1/4}\log^{1/2} n.$$

We can restrict the integration to the neighbourhood of the saddle point $(0,0,0,0)$, make a change $u = n^{-1/4}\widetilde{u}$, $t = n^{-1/4}(\pm 1 + i)\widetilde{t}_\pm$, $s = n^{-1/2}\widetilde{s}$, $r = n^{-1/2}\widetilde{r}$, then the multiplier before the integral is Cn. Estimate the multipliers in $\Phi_n(\mathbf{x})$ as follows: $\sqrt{|r - it|} = O(n^{-1/8}\log^k n)$,

$$\widetilde{\varphi}_n(u,s,t) = K_2^2 + O(n^{-1/2}\log^k n) = O(n^{-1/2}\log^k n);$$
$$u^2 I_1(2au) + \frac{a}{n} u I_0(2au) = O(n^{-1/2}\log^k n) + O(n^{-1}) = O(n^{-1/2}\log^k n).$$

Hence, $\Phi_n(\widetilde{\mathbf{x}}) \le O(n^{-1/8}\log^k n \cdot n^{-1})$ and $\varepsilon^{1/2}|T(\varepsilon,n,z)| \le Cn^{-1/8}\log^k n$, which is bounded.

Case 3. $\mathrm{tr}_n G(0) \ge 1$, $x_{0,n} \ge n^{-1/4}$. Here we use the fact that equation (3.3) has exactly one positive root $x_{0,n}$ if $\mathrm{tr}_n G(0) \ge 1$. Set $h_n(t) = -\mathcal{L}_n(-t^2) - t^2$. Similarly to Lemma 3.8, one can check that there exist constants δ, σ, C_0 and a contour $\mathbf{L}_1$ which connects points $ix_{0,n} + \delta(1 + i(1 - x_{0,n}))$ and $C_0 + iC_0$ satisfying the following conditions:

- $0 < \delta < 1$, $\sigma > 0$, $C_0 > 0$ and $\log 2C_0^2 > \mathcal{L}_n(x_{0,n}^2) + 1$;
- δ, σ, C_0 do not depend on n;

- $\mathbf{L}_1 \subset \{z \mid \Re z > 0, \Im z \ge \Re z\}$;
- $\Re h_n\big(ix_{0,n} + t(1 + i(1 - x_{0,n}))\big)$ decreases for $t \in [0, \delta]$, and $\Re h_n(t) \le \Re h_n(ix_{0,n}) - \sigma$ for $t \in \mathbf{L}_1$.

Let $\mathbf{L}_2$ be a contour symmetric to $\mathbf{L}_1$ with respect to imaginary axis. We change t-contour and r-contour as follows: $\mathbf{L}_t = \mathbf{L}_0^+ \cup \mathbf{L}_0^- \cup \mathbf{L}_1 \cup \mathbf{L}_2 \cup \mathbf{L}_1' \cup \mathbf{L}_2'$, where $\mathbf{L}_0^\pm = [ix_{0,n}, ix_{0,n} + \delta\big(\pm 1 + i(1 - x_{0,n})\big)]$, $\mathbf{L}_{1,2}' = \{(iC_0 \pm C_0 \pm \tau, \tau > 0\}$; $\widetilde{\mathcal{R}}(t) = \mathcal{R}_1(t) \cup \mathcal{R}_2(t)$, where $\mathcal{R}_1(t) = [it, -\delta]$, $\mathcal{R}_2(t) = \{r \colon r = -\delta + \rho, \rho \ge 0\}$. For (u, t, s, r) lying in a neighbourhood of $(x_{0,n}, ix_{0,n}, 0, 0)$ and $t = ix_{0,n} + (\pm 1 + i(1 - x_{0,n}))\tau$ we have

$$
\begin{aligned}
&F_{n,1}(u) = \mathcal{L}_n(x_{0,n}^2) - x_{0,n}^2 - K_1(u - x_{0,n})^2 + O(|u - x_{0,n}|^3);\\
&F_{n,2}(s, t, r) = -\mathcal{L}_n(x_{0,n}^2) + x_{0,n}^2 - K_1(t - ix_{0,n})^2 - s^2 - r^2 +\\
&+ O(|s|^3 + |t - ix_{0,n}|^3) = -\mathcal{L}_n(x_{0,n}^2) + x_{0,n}^2 - K_1(2x_{0,n} - x_{0,n}^2 + i(\ldots))\tau^2 -\\
&- s^2 - r^2 + O(|s|^3 + \tau^3)
\end{aligned}
$$

where K_1 is defined in (5.5). Since $K_1 \ge c x_{0,n}^2$ and for small enough d $\Re((2x_{0,n} - x_{0,n}^2 + i(\ldots)) \ge c x_{0,n}$, the following inequalities hold:

$$
\begin{aligned}
&n(F_{n,1}(u) - F_{n,1}(x_{0,n})) \le -C \log^2(n x_{0,n}^2), \quad |u - x_{0,n}| > (n x_{0,n}^2)^{-1/2} \log(n x_{0,n}^2);\\
&n(\Re F_{n,2}(s, t, r) - F_{n,2}(0, ix_{0,n}, 0)) \le -C \log^2(n x_{0,n}^3),\\
&\qquad\qquad\qquad\qquad |t - ix_{0,n}| > (n x_{0,n}^3)^{-1/2} \log(n x_{0,n}^3);\\
&n(\Re F_{n,2}(s, t, r) - F_{n,2}(0, ix_{0,n}, 0)) \le -C \log^2 n, \quad \max\{|s|, |r|\} > n^{-1/2} \log n.
\end{aligned}
$$

The inequalities $n x_{0,n}^2 \ge n^{1/2}$, $n x_{0,n}^3 \ge n^{1/4}$ imply that we can restrict the integration to the neighbourhood of the saddle point $u = x_{0,n}$, $t = ix_{0,n}$, $s = 0$, $r = 0$, make a change $u = x_{0,n} + (n x_{0,n}^2)^{-1/2} \widetilde{u}$, $t = ix_{0,n} + (n x_{0,n}^3)^{-1/2} \cdot (\pm 1 + i(1 - x_{0,n}))\widetilde{t}_\pm$, $s = n^{-1/2} \widetilde{s}$, $r = n^{-1/2} \widetilde{r}$, then the multiplier before the integral is $\dfrac{C n^{1/2}}{x_{0,n}^{5/2}}$. Next we estimate the multipliers in $\Phi_n(\mathbf{x})$

in the neighbourhood. Obviously, $\sqrt{|r-it|} \le C$. For $\widetilde{\varphi}(u,s,t)$ we have $\widetilde{\varphi}(x_{0,n},0,ix_{0,n}) = 0$, $\partial_u\widetilde{\varphi}(x_{0,n},0,ix_{0,n}) = O(x_{0,n}^3)$, $\partial_t\widetilde{\varphi}(x_{0,n},0,ix_{0,n}) = O(x_{0,n}^3)$, $\partial_u\partial_u\widetilde{\varphi}(x_{0,n},0,ix_{0,n}) = O(x_{0,n}^2)$, $\partial_u\partial_t\widetilde{\varphi}(x_{0,n},0,ix_{0,n}) = O(x_{0,n}^2)$, $\partial_t\partial_t\widetilde{\varphi}(x_{0,n},0,ix_{0,n}) = O(x_{0,n}^2)$. Thus

$$\begin{aligned}
|\widetilde{\varphi}(u,s,t)| &\le Cx_{0,n}^3\Big(\frac{\widetilde{u}}{(nx_{0,n}^2)^{-1/2}} + \frac{\widetilde{t}_\pm}{(nx_{0,n}^3)^{-1/2}}\Big)+ \\
&+ Cx_{0,n}^2\Big(\frac{\widetilde{u}^2}{(nx_{0,n}^2)} + \frac{\widetilde{t}_\pm^2}{(nx_{0,n}^3)}\Big) + C\Big(\frac{\widetilde{u}^3}{(nx_{0,n}^2)^{3/2}} + \frac{\widetilde{t}_\pm^3}{(nx_{0,n}^3)^{3/2}} + \frac{\widetilde{s}^2}{n}\Big) \le \\
&\le C(n^{-1/2}x_{0,n}^{3/2} + n^{-1}x_{0,n}^{-1} + n^{-3/2}x_{0,n}^{-9/2})\,\mathcal{P}(\widetilde{u},\widetilde{t}_\pm,\widetilde{s}),
\end{aligned}$$

where $\mathcal{P}(\widetilde{u},\widetilde{t}_\pm,\widetilde{s})$ is a polynomial in $\widetilde{u},\widetilde{t}_\pm,\widetilde{s}$. Next, since $I_1(2au) \le Cu$, $I_0(2au) \le C$ for small u and $(nx_{0,n}^2)^{-1/2} \le n^{-1/4} \le x_{0,n}$, we have

$$u^2 I_1(2au) + \frac{a}{n}uI_0(2au) \le Cx_{0,n}^3(1+\widetilde{u}^3).$$

This shows that $\Phi_n(\widetilde{\mathbf{x}}) \le C(n^{-1/2}x_{0,n}^{9/2} + n^{-1}x_{0,n}^2 + n^{-3/2}x_{0,n}^{-3/2})\,\widetilde{\mathcal{P}}(\widetilde{u},\widetilde{t}_\pm,\widetilde{s})$, thus $\varepsilon^{1/2}|T(\varepsilon,n,z)| \le C\Big(x_{0,n}^2 + \dfrac{1}{n^{1/2}x_{0,n}^{1/2}} + \dfrac{1}{nx_{0,n}^4}\Big)$ is bounded since $x_{0,n} \ge n^{-1/4}$.

Case 4. $\mathrm{tr}_n G(0) \ge 1$, $x_{0,n} \le n^{-1/4}$. We change t-contour and r-contour as in case 3. Denote $a_k := \mathrm{tr}_n G(x_{0,n}^k)$. Observe that $a_k \ge C_k > 0$ for each k uniformly in n and

$$\begin{aligned}
F_{n,1}(u) &= \mathcal{L}_n(u^2) - u^2 = \mathcal{L}_n(x_{0,n}^2) - x_{0,n}^2 - 2a_2x_{0,n}^2(u-x_{0,n})^2 - \\
&- (2a_2x_{0,n} - \tfrac{8}{3}a_3x_{0,n}^3)(u-x_{0,n})^3 - (\tfrac{1}{2}a_2 + O(x_{0,n}))(u-x_{0,n})^4 + O(|u-x_{0,n}|^5).
\end{aligned}$$

Studying the cases $0 \le u \le x_{0,n}$ and $x_{0,n} \le u \le C$ one can easily show that

$$F_{n,1}(u) \le \mathcal{L}_n(x_{0,n}^2) - x_{0,n}^2 - (\tfrac{1}{2}a_2 + O(x_{0,n}))(u-x_{0,n})^4 + O(|u-x_{0,n}|^5),$$

when $0 \le u \le C$ and $x_{0,n}$ is small enough (which is true when d is amsll enough). For $t = ix_{0,n} + (\pm 1 + i(1 - x_{0,n}))\tau$ we have

$$\Re F_{n,2}(s,t,r) = \mathcal{L}_n(x_{0,n}^2) - x_{0,n}^2 - 2a_2 x_{0,n}^2(2x_{0,n} + O(x_{0,n}^2))\tau^2 -$$
$$- (4a_2 x_{0,n} + O(x_{0,n}^2))\tau^3 - (2a_2 + O(x_{0,n}))\tau^4 - \widetilde{s}^2 - \widetilde{r}^2 + O(\tau^5 + |\widetilde{s}|^3) \leq$$
$$\leq \mathcal{L}_n(x_{0,n}^2) - x_{0,n}^2 - (2a_2 + O(x_{0,n}))\tau^4 - \widetilde{s}^2 - \widetilde{r}^2 + O(\tau^5 + |\widetilde{s}|^3).$$

Since $a_2 \geq C > 0$, then the following inequalities hold:

$$n(F_{n,1}(u) - F_{n,1}(x_{0,n})) \leq -C\log^2 n, \quad \text{when } |u - ix_{0,n}| > n^{-1/4}\log^{1/2} n;$$
$$n(\Re F_{n,2}(s,t,r) - F_{n,2}(0, ix_{0,n}, 0)) \leq -C\log^2 n, \quad \text{when}$$
$$\max\{|s|, |r|\} > n^{-1/2}\log n \text{ or } |t - ix_{0,n}| > n^{-1/4}\log^{1/2} n.$$

We can restrict the integration to the neighbourhood of the saddle point, make a change $u = x_{0,n} + n^{-1/4}\widetilde{u}$, $t = ix_{0,n} + n^{-1/4}(\pm 1 + i(1 - x_{0,n}))\widetilde{t}_\pm$, $s = n^{-1/2}\widetilde{s}$, $r = n^{-1/2}\widetilde{r}$, then the multiplier before the integral is Cn. Estimate the multipliers in $\Phi_n(\mathbf{x})$ as follows: $\sqrt{|r - it|} \leq C$;

$$\widetilde{\varphi}_n(u,s,t) \leq C(n^{-1/4}\widetilde{u} + n^{-1/4}\widetilde{t}_\pm + n^{-1}\widetilde{s}^2);$$
$$u^2 I_1(2au) + \frac{a}{n}uI_0(2au) \leq C(x_{0,n} + n^{-1/4}\widetilde{u})^3 + \frac{C}{n}(x_{0,n} + n^{-1/4}\widetilde{u}) \leq Cn^{-3/4}\mathcal{P}(\widetilde{u}).$$

Hence, $\Phi_n(\widetilde{\mathbf{x}}) \leq Cn^{-1}\widetilde{\mathcal{P}}(\widetilde{u}, \widetilde{t}_\pm, \widetilde{s})$, and $\varepsilon^{1/2}|T(\varepsilon, n, z)|$ is bounded.

□

5.2. Case $\varepsilon > a/n$

Proposition 5.6. *Fix an arbitrary $d > 0$, and set $E_{d,in} = \{z \in E \mid z \in D$, $\operatorname{dist}(z, \partial D) \geq d\}$. Then there exist $n_0 \in \mathbb{N}$, $a > 0$ and $C > 0$ such that $\varepsilon^{1/2}|T(\varepsilon, n, z)| \leq C$ for $n \geq n_0$, $\dfrac{a}{n} < \varepsilon < \varepsilon_0$ and $z \in E_{d,in}$.*

Proof. Since $E_{d,in}$ is a compact subset of $\operatorname{Int} D$, we can use the results of Section 3.1 for $E_{in} = E_{d,in}$. We want to prove that $\varepsilon^{1/2}|T(\varepsilon, n, z)| = O(1)$ as

$n \to \infty$ and $n\varepsilon \to \infty$.

Make the same change of variables as in Section 3.2: $r = R + it + \varepsilon \in \mathcal{R}(t)$, where $\mathcal{R}(t) = \{-it - \varepsilon + \tau,\ \tau > 0\}$, $u = \sqrt{u_1^2 + u_2^2} \in [0, +\infty)$, $w = \sqrt{u_1 + u} \in [0, \sqrt{2u}]$. We obtain

$$\varepsilon^{1/2} T(\varepsilon, n, z) = \frac{16n^3\sqrt{\varepsilon}}{\pi^3} \int_{\mathcal{V}} \Phi_n(\mathbf{x}) \exp\{n\, F_n(\mathbf{x})\}\, d\mathbf{x},$$

where

$$\begin{aligned}
&\mathbf{x} = (u, s, t, r, v, w), \quad \mathcal{V} = [0, +\infty) \times \mathbb{R} \times L \times \mathcal{R}(t) \times \mathbb{R} \times [0, \sqrt{2u}],\\
&F_n(\mathbf{x}) = F_{n,1}(u) + F_{n,2}(s, t, r) - \varepsilon v^2 - 2\varepsilon w^2,\\
&F_{n,1}(u) = \mathcal{L}_n(u^2) - (u - \varepsilon)^2,\ F_{n,2}(s, t, r) = -\mathcal{L}_n(s^2 - t^2) - (t - i\varepsilon)^2 - r^2,\\
&\widetilde{\varphi}_n(u, s, t) = \varphi_n(u^2, s^2 - t^2, z, z),\\
&\Phi_n(\mathbf{x}) = \frac{r - it - \varepsilon}{\sqrt{v^2 + 4(r - it - \varepsilon)}} \cdot \frac{u}{\sqrt{2u - w^2}} \cdot \widetilde{\varphi}_n(u, s, t) \cdot (u - w^2 - \varepsilon).
\end{aligned} \tag{5.11}$$

As in Section 3.2, we can change t-contour, r-contour and expand $F_{n,1}$ and $F_{n,2}$ as follows:

$$\begin{aligned}
F_{n,1}(u) &= \mathcal{L}_n(x_{\varepsilon,n}^2) - (x_{\varepsilon,n} - \varepsilon)^2 - \kappa_1(u - x_{\varepsilon,n})^2 + O\big((u - x_{\varepsilon,n})^3\big),\\
F_{n,2}(s, t, r) &= -\mathcal{L}_n(x_{\varepsilon,n}^2) + (x_{\varepsilon,n} - \varepsilon)^2 - \kappa_1(t - ix_{\varepsilon,n})^2 - \kappa_3 s^2 - r^2 +\\
&\qquad + O(|t - ix_{\varepsilon,n}|^3 + |s|^3),
\end{aligned} \tag{5.12}$$

where $\kappa_{1,3}$ are defined in (3.7), (3.20). Proposition 3.2 shows that $\kappa_1, \kappa_3 \geq C > 0$, then we can make the following estimations:

$$\begin{aligned}
&F_{n,1}(u) \leq F_{n,1}(x_{\varepsilon,n}) - Cn^{-1}\log^2 n \quad \text{when } |u - x_{\varepsilon,n}| > n^{-1/2}\log n;\\
&\Re F_{n,2}(s, t, r) \leq \Re F_{n,2}(0, ix_{\varepsilon,n}, 0) - Cn^{-1}\log^2 n \quad \text{when}\\
&\qquad\qquad \max\{|s|, |t - ix_{\varepsilon,n}|, |r|\} > n^{-1/2}\log n.
\end{aligned} \tag{5.13}$$

Now split the integration area into three parts:

$$\widetilde{\mathcal{V}} = \mathcal{V}_1 \sqcup \mathcal{V}_2 \sqcup \mathcal{V}_3,$$
$$\mathcal{V}_1 := \{\mathbf{x} \in \widetilde{\mathcal{V}} \colon |v|, |w| < (n\varepsilon)^{-1/2} \log(n\varepsilon),\ |u|, |s|, |t|, |r| < n^{-1/2} \log n\},$$
$$\mathcal{V}_2 := \{\mathbf{x} \in \widetilde{\mathcal{V}} \colon \max\{|v|, |w|\} > (n\varepsilon)^{-1/2} \log(n\varepsilon),\ |u|, |s|, |t|, |r| < n^{-1/2} \log n\},$$
$$\mathcal{V}_3 := \{\mathbf{x} \in \widetilde{\mathcal{V}} \colon \max\{|u|, |s|, |t|, |r|\} > n^{-1/2} \log n\}$$

Then we can write

$$\varepsilon^{1/2} T(\varepsilon, n, z) = \Big(\frac{Cn^2(n\varepsilon)}{\varepsilon^{1/2}} \int_{\mathcal{V}_1} + \frac{Cn^2(n\varepsilon)}{\varepsilon^{1/2}} \int_{\mathcal{V}_2} + Cn^3 \sqrt{\varepsilon} \int_{\mathcal{V}_3} \Big) \Phi_n(\mathbf{x})\, e^{nF_n(\mathbf{x})}\, d\mathbf{x}.$$

Integration over $\mathcal{V}_3$. Note that for $n \geq 2n_0$ we have

$$\Big| \frac{4n^3\sqrt{\varepsilon}}{\pi^3} \int_{\mathcal{V}_3} \Big| \leq \frac{4n^3\sqrt{\varepsilon}}{\pi^3} \cdot e^{-C\log^2 n} \int_{\widetilde{\mathcal{V}}} |\Phi_n(\mathbf{x})| e^{n_0 \Re F_n(\mathbf{x})}\, d\mathbf{x}.$$

One can check that the integral above is bounded uniformly in n, ε.

Integration over $\mathcal{V}_2$. Make a change of variables $u = x_{\varepsilon,n} + n^{-1/2}\widetilde{u}$, $s = n^{-1/2}\widetilde{s}$, $t = ix_{\varepsilon,n} + n^{-1/2}\widetilde{t}$, $r = n^{-1/2}\widetilde{r}$. Notice that $-(n\varepsilon)v^2 - 2(n\varepsilon)w^2 \leq -v^2 - 2w^2 - C\log^2(n\varepsilon)$ when $\max\{|v|, |w|\} > (n\varepsilon)^{-1/2}\log(n\varepsilon)$ and $n\varepsilon \geq 2$, thus

$$\Big| \frac{Cn^2(n\varepsilon)}{\varepsilon^{1/2}} \int_{\mathcal{V}_2} \Big| \leq \frac{C(n\varepsilon)\, e^{-C\log^2(n\varepsilon)}}{\varepsilon^{1/2}} \int |\Phi_n(\mathrm{x})| \times$$
$$\times \exp\{n(F_{n,1}(u) + \Re F_{n,2}(s,t,r)) - v^2 - 2w^2\}\, d\widetilde{u}\, d\widetilde{s}\, d\widetilde{t}\, d\widetilde{r}\, dv\, dw.$$

Recall the definition of Φ_n and the inequality $0 \leq w^2 \leq 2u$, then

$$|\Phi_n(\mathbf{x})| \leq C\sqrt{|r - it - \varepsilon|} \cdot \frac{u}{\sqrt{2u - w^2}} \cdot (u + \varepsilon) \cdot |\widetilde{\varphi}_n(u,s,t)|.$$

Using this estimation and integrating over v, w, we obtain

$$\Big|\frac{Cn^2(n\varepsilon)}{\varepsilon^{1/2}}\int_{\mathcal{V}_2}\Big| \le \frac{C(n\varepsilon)\, e^{-C\log^2(n\varepsilon)}}{\varepsilon^{1/2}}\int \sqrt{|r-it-\varepsilon|}\cdot u(u+\varepsilon)\cdot e^{-2u}I_0(2u)\times$$
$$\times\, |\widetilde{\varphi}_n(u,s,t)|\cdot \exp\{n(F_{n,1}(u)+F_{n,2}(s,t,r))\}\, d\widetilde{u}\, d\widetilde{s}\, d\widetilde{t}\, d\widetilde{r} \le$$
$$\le \frac{C(n\varepsilon)\, e^{-C\log^2(n\varepsilon)}}{\varepsilon^{1/2}}\int |\widetilde{\varphi}_n(u,s,t)|\, \exp\{n(F_{n,1}(u)+F_{n,2}(s,t,r))\}\, d\widetilde{u}\, d\widetilde{s}\, d\widetilde{t}\, d\widetilde{r}$$

Next we expand $\widetilde{\varphi}_n(u,s,t)$ in the neighbourhood of $(x_{\varepsilon,n}, 0, ix_{\varepsilon,n})$:

$$\widetilde{\varphi}_n(u,s,t) = \widetilde{\varphi}_n(x_{\varepsilon,n},0,ix_{\varepsilon,n}) + n^{-1/2}(\phi_u\widetilde{u}+\phi_t\widetilde{t}) + O(n^{-1}\log^2 n), \tag{5.14}$$

where $\phi_u = \partial_u\widetilde{\varphi}_n(x_{\varepsilon,n},0,ix_{\varepsilon,n})$, $\phi_t = \partial_t\widetilde{\varphi}_n(x_{\varepsilon,n},0,ix_{\varepsilon,n})$ are bounded. Remark 2.1 gives us $\widetilde{\varphi}_n(x_{\varepsilon,n},0,ix_{\varepsilon,n}) = \dfrac{\varepsilon^2}{x^2_{\varepsilon,n}} + 2\varepsilon x_{\varepsilon,n}\cdot \mathrm{tr}_n G^2(x^2_{\varepsilon,n}) = O(\varepsilon)$. Finally, using expansion (5.12), we obtain

$$\begin{aligned}\Big|\frac{Cn^2(n\varepsilon)}{\varepsilon^{1/2}}\int_{\mathcal{V}_2}\Big| &\le C(n\varepsilon)\, e^{-C\log^2(n\varepsilon)}\cdot\varepsilon^{-1/2}\int d\widetilde{u}\, d\widetilde{s}\, d\widetilde{t}\, d\widetilde{r}\times\\ &\times \Big(O(\varepsilon)+\frac{|\phi_u||\widetilde{u}|+|\phi_t||\widetilde{t}|}{n^{1/2}}+O(n^{-1}\log^2 n)\Big)\, e^{-\frac{\kappa_1}{2}\widetilde{u}^2-\frac{\kappa_1}{2}\widetilde{t}^2-\frac{\kappa_3}{2}\widetilde{s}^2-\widetilde{r}^2} =\\ &= C(n\varepsilon)\, e^{-C\log^2(n\varepsilon)}\Big(O(\varepsilon^{1/2})+O((n\varepsilon)^{-1/2})\Big),\end{aligned} \tag{5.15}$$

which is bounded uniformly in $\varepsilon\le\varepsilon_0$, $n\ge n_0$, $n\varepsilon\ge a$.

Integration over $\mathcal{V}_1$. Make a change of variables $u = x_{\varepsilon,n}+\widetilde{u}_1 n^{-1/2}$, $t = ix_{\varepsilon,n}+\widetilde{t}n^{-1/2}$, $s=\widetilde{s}n^{-1/2}$, $r=\widetilde{r}n^{-1/2}$, $v=\widetilde{v}(n\varepsilon)^{-1/2}$, $w=\widetilde{w}(n\varepsilon)^{-1/2}$. According to Section 3.1, $x_{\varepsilon,n}\ge\kappa_0>0$, thus $\sqrt{2u-w^2}\ge C>0$ in the neighbourhood. Then $|\Phi_n(\mathrm{x})|\le C\cdot|\widetilde{\varphi}_n(u,s,t)|$ and we can expand $\widetilde{\varphi}_n$ as in (5.14). We obtain

$$\Big|\frac{Cn^2(n\varepsilon)}{\varepsilon^{1/2}}\int_{\mathcal{V}_1}\Big| \le \varepsilon^{-1/2}\int \big(O(\varepsilon)+n^{-1/2}(|\phi_u||\widetilde{u}|+|\phi_t||\widetilde{t}|)+O(n^{-1}\log^2 n)\big)\times$$
$$\times\, e^{-\frac{\kappa_1}{2}\widetilde{u}^2-\frac{\kappa_1}{2}\widetilde{t}^2-\frac{\kappa_3}{2}\widetilde{s}^2-\widetilde{r}^2-\widetilde{v}^2-2\widetilde{w}^2}\, d\widetilde{u}\, d\widetilde{s}\, d\widetilde{t}\, d\widetilde{r}\, d\widetilde{v}\, d\widetilde{w} = O(\varepsilon^{1/2})+O((n\varepsilon)^{-1/2})$$

which is bounded uniformly in $\varepsilon\le\varepsilon_0$, $n\ge n_0$, $n\varepsilon\ge a$.

$\square$

Proposition 5.7. *Fix an arbitrary $d > 0$, and set $E_{d,out} := \{z \in E \mid z \notin D,\ \mathrm{dist}(z, \partial D) \geq d\}$. Then there exist $n_0 \in \mathbb{N}$, $a > 0$ and $C > 0$ such that $\varepsilon^{1/2} |T(\varepsilon, n, z)| \leq C$ for $n \geq n_0$, $\dfrac{a}{n} < \varepsilon < \varepsilon_0$ and $z \in E_{d,out}$.*

Proof. Introduce the averaging:

$$\begin{aligned}
&\langle\!\langle f(u_1)\rangle\!\rangle = \frac{2n^3}{\pi^3}\int_0^\infty dR \int_{-\infty}^{\infty} dv\, du_1\, du_2\, ds \int_L dt \cdot \frac{R}{\sqrt{v^2+4R}} \cdot f(u_1) \times \\
&\times \varphi_n(u_1^2{+}u_2^2, s^2-t^2, z, z) \cdot \exp\Big\{n\Big(\mathcal{L}_n(u_1^2{+}u_2^2) - (u_1+\varepsilon)^2 - u_2^2\Big)\Big\} \times \\
&\times \exp\Big\{-n\Big(\mathcal{L}_n(s^2-t^2) + (t-i\varepsilon)^2 + (R+it+\varepsilon)^2 + \varepsilon v^2\Big)\Big\},
\end{aligned} \tag{5.16}$$

Then

$$\varepsilon^{1/2}\, T(\varepsilon, n, z) = -2\varepsilon^{1/2}\langle\!\langle u_1+\varepsilon\rangle\!\rangle = 2\varepsilon^{1/2}(x_{\varepsilon,n}-\varepsilon)\langle\!\langle 1\rangle\!\rangle - 2\varepsilon^{1/2}\langle\!\langle u_1 + x_{\varepsilon,n}\rangle\!\rangle$$

Identity (2.1) implies that $\langle\!\langle 1\rangle\!\rangle = \mathcal{Z}(\varepsilon,\varepsilon,z,z) = 1$, hence it suffices to prove that $\varepsilon^{1/2}\big|\, \langle\!\langle u_1 + x_{\varepsilon,n}\rangle\!\rangle\, \big|$ is bounded. Having $\varepsilon^{1/2}\langle\!\langle u_1 + x_{\varepsilon,n}\rangle\!\rangle$ one can change t-contour and r-contour as in Section 3.3. For $t \in \mathbf{L}_t$, $r \in \widetilde{\mathcal{R}}(t)$ we have $\Big|\dfrac{r-it-\varepsilon}{\sqrt{v^2+4(r-it-\varepsilon}}\Big| \leq \dfrac{1}{2}\sqrt{|r-it-\varepsilon|}$. Next we integrate with respect to v and obtain $\varepsilon^{1/2}\big|\langle\!\langle u_1 + x_{\varepsilon,n}\rangle\!\rangle\big| \leq Cn^{5/2}\displaystyle\int_{\mathcal{V}} \widehat{\Phi}_n(\mathbf{x}) \exp\{n\,\widehat{F}_n(\mathbf{x})\}\, d\mathbf{x}$, where

$$\begin{aligned}
&F_{n,1}(u_1,u_2) = \mathcal{L}_n(u_1^2+u_2^2) - (u_1+\varepsilon)^2 - u_2^2,\\
&F_{n,2}(s,t,r) = -\Big(\mathcal{L}_n(s^2-t^2) + (t-i\varepsilon)^2 + r^2\Big),\\
&\widehat{F}_n(\mathbf{x}) = F_{n,1}(u_1,u_2) + F_{n,2}(s,t,r),\\
&\widetilde{\varphi}_n(u_1,u_2,s,t) = \varphi_n(u_1^2+u_2^2, s^2-t^2, z, z),\\
&\widehat{\Phi}_n(\mathbf{x}) = |u_1 + x_{\varepsilon,n}|\, \sqrt{|r-it-\varepsilon|} \cdot |\widetilde{\varphi}_n(u_1,u_2,s,t)|.
\end{aligned} \tag{5.17}$$

We have the following expansions:

$$F_{n,1}(u_1,u_2) = \mathcal{L}_n(x_{\varepsilon,n}^2) - (x_{\varepsilon,n}-\varepsilon)^2 - \kappa_1(u+x_{\varepsilon,n})^2 - \kappa_2 u_2^2 + O(|u_1+x_{\varepsilon,n}|^3+|u_2|^3);$$
$$F_{n,2}(s,t,r) = -\mathcal{L}_n(x_{\varepsilon,n}^2) + (x_{\varepsilon,n}-\varepsilon)^2 - (\kappa_3 s^2 + \kappa_4\tau^2 + \kappa_5\tau^3 + \kappa_6\tau^4 + r^2) +$$
$$+ O(|s|^3 + \tau^5),$$

where $t = ix_{\varepsilon,n} + (\pm 1 + i(1+\varepsilon - x_{\varepsilon,n}))\tau$, κ_1, κ_2, κ_3, κ_4 are defined in (3.7), (3.22), (3.20), (3.23), and

$$\Re\kappa_5 = 4x_{\varepsilon,n}\cdot \mathrm{tr}_n G^2(x_{\varepsilon,n}^2) + O(x_{\varepsilon,n}^2), \quad \kappa_6 = 2\cdot \mathrm{tr}_n G^2(x_{\varepsilon,n}^2) + O(x_{\varepsilon,n}^2). \quad (5.18)$$

Proposition 3.4 shows that $\kappa_{1,2,3} \geq C > 0$, $\kappa_{4,5} \geq C\varepsilon > 0$, $\kappa_6 \geq C > 0$. We obtain the following inequalities:

$$F_{n,1}(u_1,u_2) \leq F_{n,1}(-x_{\varepsilon,n},0) - Cn^{-1}\log^2 n, \ \max\{|u_1 - x_{\varepsilon,n}|, |u_2|\} > n^{-1/2}\log n;$$
$$\Re F_{n,2}(s,t,r) \leq F_{n,2}(0, ix_{\varepsilon,n}, 0) - Cn^{-1}\log^2 n, \quad \text{when}$$
$$\max\{|s|,|r|\} > n^{-1/2}\log n \text{ or } |t - ix_{\varepsilon,n}| > n^{-1/4}\log^{1/2} n.$$

This shows that we can restrict the integration to the neighbourhood. Make a change $u_1 = -x_{\varepsilon,n} + n^{-1/2}\widetilde{u}_1$, $u_2 = n^{-1/2}\widetilde{u}_2$, $s = n^{-1/2}\widetilde{s}$, $r = n^{-1/2}\widetilde{r}$, $t = ix_{\varepsilon,n} + n^{-1/4}(1 + i(1+\varepsilon - x_{\varepsilon,n}))\widetilde{t}_\pm$ and notice that the multiplier $|u_1 + x_{\varepsilon,n}|$ in $\widehat{\Phi}_n$ is equal to $n^{-1/2}|\widetilde{u}_1|$, thus

$$\Big|Cn^{5/2}\int_{\mathcal{V}_1} \widehat{\Phi}_n(\mathbf{x})\exp\{n\widehat{F}_n(\mathbf{x})\}\,d\mathbf{x}\Big| \leq Cn^{-1/4}\times$$
$$\times \int |\widetilde{u}_1|\,\exp\{-\tfrac{\kappa_1}{2}\widetilde{u}_1^2 - \tfrac{\kappa_2}{2}\widetilde{u}_2^2 - \tfrac{\kappa_3}{2}\widetilde{s}^2 - \tfrac{\kappa_6}{2}\widetilde{t}_\pm^4 - \widetilde{r}^2\}\,d\widetilde{u}_1\,d\widetilde{u}_2\,d\widetilde{s}\,d\widetilde{t}_\pm\,d\widetilde{r}\,d\widetilde{v},$$

and the integral is bounded. $\square$

Proposition 5.8. *There exist $d > 0$, $n_0 \in \mathbb{N}$, $a > 0$ and $C > 0$ such that $\varepsilon^{3/4}|T(\varepsilon,n,z)| \leq C$ for $n \geq n_0$, $\dfrac{a}{n} < \varepsilon < \varepsilon_0$ and $z \in E_{d,\partial}$, where $E_{d,\partial} = \{z \in E \mid \mathrm{dist}(z, \partial D) \leq d\}$.*

Proof. We split the proof into two cases:

Case 1. $\mathrm{tr}_n G(0) \geq 1$. Let $x_{\varepsilon,n}$ be the positive root of (3.1). According to Section 3.1, we have $c\varepsilon^{1/3} \leq x_{\varepsilon,n} \leq C$ for some $c, C > 0$. As in Proposition 5.6 we obtain $\varepsilon^{3/4} T(\varepsilon, n, z) = \dfrac{16n^3\varepsilon^{3/4}}{\pi^3} \displaystyle\int_{\mathcal{V}} \Phi_n(\mathbf{x}) \exp\{n\, F_n(\mathbf{x})\}\, d\mathbf{x}$, with the same notations as in (5.11). Change t-contour and r-contour as in Section 3.3 and make the estimations: $\left| \dfrac{r - it - \varepsilon}{\sqrt{v^2 + 4(r - it - \varepsilon)}} \right| \leq \sqrt{|r - it - \varepsilon|}$; $\dfrac{u}{\sqrt{2u - w^2}} \cdot (u - w^2 - \varepsilon) \leq u(u + \varepsilon) \cdot \dfrac{1}{\sqrt{2u - w^2}}$. Now we can integrate with respect to v, w. One can show that

$$\int e^{-n\varepsilon v^2}\, dv = \sqrt{\frac{\pi}{n\varepsilon}}; \qquad \int_0^{\sqrt{2u}} \frac{e^{-2n\varepsilon w^2}}{\sqrt{2u - w^2}}\, dw = \frac{\pi}{2} e^{-2n\varepsilon u} I_0(2n\varepsilon u),$$

where $I_0(x)$ is a modified Bessel function. Asymptotic formulas for $I_0(x)$ imply that $e^{-x} I_0(x) \leq \dfrac{C}{\sqrt{x}}$ for all $x \geq 0$ and some $C > 0$. We can apply the inequality $e^{-2n\varepsilon u} I_0(2n\varepsilon u) \leq \dfrac{C}{\sqrt{2n\varepsilon u}}$ to obtain

$$\varepsilon^{3/4} \left| T(\varepsilon, n, z) \right| \leq C n^2 \varepsilon^{-1/4} \int_{\mathcal{V}} \widehat{\Phi}_n(\mathbf{x}) \exp\{n\, \widehat{F}_n(\mathbf{x})\}\, d\mathbf{x}, \tag{5.19}$$

where

$$\begin{aligned}
&\mathbf{x} = (u, s, t, r), \quad \mathcal{V} = [0, +\infty) \times \mathbb{R} \times L \times \mathcal{R}(t), \\
&F_{n,1}(u) = \mathcal{L}_n(u^2) - (u - \varepsilon)^2, \quad F_{n,2}(s, t, r) = -\mathcal{L}_n(s^2 - t^2) - (t - i\varepsilon)^2 - r^2, \\
&\widehat{F}_n(\mathbf{x}) = F_{n,1}(u) + F_{n,2}(s, t, r), \quad \widetilde{\varphi}_n(u, s, t) = \varphi_n(u^2, s^2 - t^2, z, z), \\
&\widehat{\Phi}_n(\mathbf{x}) = \sqrt{|r - it - \varepsilon|} \cdot \sqrt{u}(u + \varepsilon) \cdot |\widetilde{\varphi}_n(u, s, t)|.
\end{aligned}$$

Consider the following subcases:

Subcase 1a. Suppose $\varepsilon^{1/4} \leq x_{\varepsilon,n} \leq C$. For (u, t, s, r) lying in a neighbourhood of $(x_{\varepsilon,n}, ix_{\varepsilon,n}, 0, 0)$ and $t = ix_{\varepsilon,n} + (\pm 1 + i(1+\varepsilon - x_{\varepsilon,n}))\tau$ we have

$$F_{n,1}(u) = \mathcal{L}_n(x_{\varepsilon,n}^2) - (x_{\varepsilon,n} - \varepsilon)^2 - \kappa_1(u - x_{\varepsilon,n})^2 + O((u - x_{\varepsilon,n})^3);$$
$$F_{n,2}(s,t,r) = -\mathcal{L}_n(x_{\varepsilon,n}^2) + (x_{\varepsilon,n} - \varepsilon)^2 - \kappa_4\tau^2 - \kappa_5\tau^3 - \kappa_6\tau^4 - \kappa_3 s^2 - r^2 +$$
$$+ O(|s|^3 + |t - ix_{\varepsilon,n}|^5).$$

where $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ are defined in (3.7), (3.22), (3.20), (3.23) and κ_5, κ_6 satisfy (5.18). It is easy to see that $\kappa_1 \geq Cx_{\varepsilon,n}^2$, $\kappa_3 \geq C$, $\kappa_4 \geq Cx_{\varepsilon,n}^3$. Thus the following inequalities hold:

$$nF_{n,1}(u) \leq n\mathcal{F}_{n,1}(x_{\varepsilon,n}) - C\log^2(nx_{\varepsilon,n}^2), \quad |u - x_{\varepsilon,n}| > (nx_{\varepsilon,n}^2)^{-1/2}\log(nx_{\varepsilon,n}^2);$$
$$n\Re F_{n,2}(s,t,r) \leq n\mathcal{F}_{n,2}(0, ix_{\varepsilon,n}, 0) - C\log^2(nx_{\varepsilon,n}^3),$$
$$|t - ix_{\varepsilon,n}| > (nx_{\varepsilon,n}^3)^{-1/2}\log(nx_{\varepsilon,n}^3);$$
$$n\Re F_{n,2}(s,t,r) \leq n\mathcal{F}_{n,2}(0, ix_{\varepsilon,n}, 0) - C\log^2 n, \quad \max\{|s|,|r|\} > n^{-1/2}\log n.$$

Since $nx_{\varepsilon,n}^2 \geq n\varepsilon^{1/2} \geq cn^{1/2}$ and $nx_{\varepsilon,n}^3 \geq n\varepsilon^{3/4} \geq cn^{1/4}$, the above bounds allow to restrict the integration to the neighbourhood of the saddle point. Make a change $u = x_{\varepsilon,n} + (nx_{\varepsilon,n}^2)^{-1/2}\widetilde{u}$, $t = ix_{\varepsilon,n} + (nx_{\varepsilon,n}^3)^{-1/2}(\pm 1 + i(1 + \varepsilon - x_{\varepsilon,n}))\widetilde{t}_\pm$, $s = n^{-1/2}\widetilde{s}$, $r = n^{-1/2}\widetilde{r}$, then the multiplier before the integral is $\dfrac{C}{\varepsilon^{1/4}x_{\varepsilon,n}^{5/2}}$. One can compute the derivatives of $\widetilde{\varphi}_n(u,t,s)$ and check that $\widetilde{\varphi}_n(x_{\varepsilon,n}, ix_{\varepsilon,n}, 0) \leq C\varepsilon x_{\varepsilon,n}$, the derivatives of the first order at $(x_{\varepsilon,n}, ix_{\varepsilon,n}, 0)$ are $O(x_{\varepsilon,n}^3)$, the derivatives of the second order at $(x_{\varepsilon,n}, ix_{\varepsilon,n}, 0)$ are $O(x_{\varepsilon,n}^2)$, the derivatives of the third order at $(x_{\varepsilon,n}, ix_{\varepsilon,n}, 0)$ are $O(x_{\varepsilon,n})$ and the derivatives of the fourth order and higher at $(x_{\varepsilon,n}, ix_{\varepsilon,n}, 0)$ are $O(1)$. Thus

$$|\widetilde{\varphi}_n(u,t,s)| \leq \Big(\varepsilon x_{\varepsilon,n} + \frac{x_{\varepsilon,n}^3}{(nx_{\varepsilon,n}^3)^{1/2}} + \frac{x_{\varepsilon,n}^2}{(nx_{\varepsilon,n}^3)} + \frac{x_{\varepsilon,n}}{(nx_{\varepsilon,n}^3)^{3/2}} + \frac{1}{(nx_{\varepsilon,n}^3)^2} + \frac{1}{n^{1/2}}\Big)\times$$
$$\times\,\mathcal{P}(\widetilde{u}, \widetilde{t}_\pm, \widetilde{s}),$$

where $\mathcal{P}(\widetilde{u}, \widetilde{t}_{\pm}, \widetilde{s})$ is a polynomial. The inequality $\dfrac{1}{(nx_{\varepsilon,n}^2)^{1/2}} \leq \dfrac{C\varepsilon^{1/2}}{x_{\varepsilon,n}} \leq Cx_{\varepsilon,n}$ shows that $u = x_{\varepsilon,n} + (nx_{\varepsilon,n}^2)^{-1/2}\widetilde{u} \leq Cx_{\varepsilon,n}(1 + |\widetilde{u}|)$. Using the bounds on $\widetilde{\varphi}$, u and the fact that $\dfrac{a}{n} < \varepsilon$, we obtain:

$$\varepsilon^{3/4}|T(\varepsilon, n, z)| \leq C(\varepsilon^{3/4} + \varepsilon^{1/4}x_{\varepsilon,n}^{1/2} + \varepsilon^{3/4}x_{\varepsilon,n}^{-2} + \varepsilon^{5/4}x_{\varepsilon,n}^{-9/2} + \varepsilon^{7/4}x_{\varepsilon,n}^{-7} + \varepsilon^{1/4}x_{\varepsilon,n}^{-1}),$$

which is bounded, since $\varepsilon^{1/4} \leq x_{\varepsilon,n} \leq C$.

Subcase 1b. Suppose that $\varepsilon > n^{-1/2}$. Then the inequality $x_{\varepsilon,n} \geq c\varepsilon^{1/3} > cn^{-1/6}$ shows that $nx_{\varepsilon,n}^2 > cn^{2/3}$ and $nx_{\varepsilon,n}^3 > cn^{1/2}$. Then we can restrict the integration to the same neighbourhood as in Subcase 1a and make the same change of variables. Moreover, $\dfrac{1}{(nx_{\varepsilon,n}^2)^{1/2}} \leq \dfrac{C\varepsilon}{x_{\varepsilon,n}} \leq Cx_{\varepsilon,n}$. Using $\varepsilon > n^{-1/2} \Leftrightarrow n^{-1} < \varepsilon^2$, we get

$$\varepsilon^{3/4}|T(\varepsilon, n, z)| \leq C(\varepsilon^{3/4} + \varepsilon^{3/4}x_{\varepsilon,n}^{1/2} + \varepsilon^{7/4}x_{\varepsilon,n}^{-2} + \varepsilon^{11/4}x_{\varepsilon,n}^{-9/2} + \varepsilon^{15/4}x_{\varepsilon,n}^{-7} + \varepsilon^{3/4}x_{\varepsilon,n}^{-1}),$$

which is bounded since $c\varepsilon^{1/3} \leq x_{\varepsilon,n} \leq C$.

Subcase 1c. Suppose $\dfrac{a}{n} < \varepsilon < \dfrac{1}{n^{1/2}}$, $c\varepsilon^{1/3} \leq x_{\varepsilon,n} \leq \varepsilon^{1/4}$. For $t = ix_{\varepsilon,n} + (\pm 1 + i(1 + \varepsilon - x_{\varepsilon,n}))\tau$ we have

$$\Re F_{n,2}(s, t, r) \leq -\mathcal{L}_n(x_{\varepsilon,n}^2) + (x_{\varepsilon,n} - \varepsilon)^2 - \Re\kappa_6\tau^4 - \kappa_3 s^2 - r^2 + O(|s|^3 + \tau^5).$$

Since $\kappa_1 \geq Cx_{\varepsilon,n}^2$, $\kappa_3, \Re\kappa_6 \geq C > 0$, then the following inequalities hold:

$$\begin{aligned} nF_{n,1}(u) &\leq n\mathcal{F}_{n,1}(x_{\varepsilon,n}) - C\log^2(nx_{\varepsilon,n}^2), \quad |u - x_{\varepsilon,n}| > (nx_{\varepsilon,n}^2)^{-1/2}\log(nx_{\varepsilon,n}^2); \\ n\Re F_{n,2}(s, t, r) &\leq n\mathcal{F}_{n,2} - C\log^2 n, \quad \max\{|s|, |r|\} > n^{-1/2}\log n \text{ or} \\ &\qquad\qquad |t - ix_{\varepsilon,n}| > n^{-1/4}\log^{1/2} n. \end{aligned}$$

We have $nx_{\varepsilon,n}^2 \geq n\varepsilon^{2/3} \geq cn^{1/3}$, thus the above bounds allow to restrict the integration to the neighbourhood of the saddle point. Make a change $u = x_{\varepsilon,n} + (nx_{\varepsilon,n}^2)^{-1/2}\widetilde{u}$, $t = ix_{\varepsilon,n} + n^{-1/4}(\pm 1 + i(1+\varepsilon - x_{\varepsilon,n}))\widetilde{t}_\pm$, $s = n^{-1/2}\widetilde{s}$, $r = n^{-1/2}\widetilde{r}$, then the multiplier before the integral is $\dfrac{Cn^{1/4}}{\varepsilon^{1/4}x_{\varepsilon,n}}$. An argument similar to the one in Subcase 1a allows to make the following estimation:

$$|\widetilde{\varphi}_n(u,t,s)| \le C\Big(\varepsilon x_{\varepsilon,n} + \frac{x_{\varepsilon,n}^3}{n^{1/4}} + \frac{x_{\varepsilon,n}^2}{n^{1/2}} + \frac{1}{n} + \frac{x_{\varepsilon,n}}{n^{3/4}} + \frac{x_{\varepsilon,n}}{(nx_{\varepsilon,n}^2)^{3/2}} + \frac{1}{(nx_{\varepsilon,n}^2)^2}\Big)\times \\ \times \mathcal{P}(\widetilde{u},\widetilde{t}_\pm,\widetilde{s})$$

where $\mathcal{P}(\widetilde{u},\widetilde{t}_\pm,\widetilde{s})$ is a polynomial. We also know that $\sqrt{u}(u+\varepsilon) \le C$ for u lying in the neighbourhood of $x_{\varepsilon,n}$. Using $\dfrac{a}{n} < \varepsilon$, we obtain:

$$\varepsilon^{3/4}|T(\varepsilon,n,z)| \le C\Big(n^{1/4}\varepsilon^{3/4} + \frac{x_{\varepsilon,n}^2}{\varepsilon^{1/4}} + x_{\varepsilon,n} + \frac{\varepsilon^{1/2}}{x_{\varepsilon,n}} + \varepsilon^{1/4} + \frac{\varepsilon}{x_{\varepsilon,n}^3} + \frac{\varepsilon^{3/2}}{x_{\varepsilon,n}^{9/2}}\Big),$$

which is bounded, since $c\varepsilon^{1/3} \le x_{\varepsilon,n} \le x_{\varepsilon,n}^{1/4}$ and $\varepsilon < \dfrac{1}{n^{1/2}} \Leftrightarrow n^{1/4} < \dfrac{1}{\varepsilon^{1/2}}$.

Case 2. $\mathrm{tr}_n G(0) < 1$. Let $x_{\varepsilon,n}$ be the positive root of (3.1). According to Proposition 3.5, we have $(1+c)\varepsilon \le x_{\varepsilon,n} \le C\varepsilon^{1/3}$ for some $c, C > 0$. Similarly to Proposition 5.7 we define $\langle\!\langle f(u_1,u_2,t,s,R,v)\rangle\!\rangle$ and make the estimation

$$\varepsilon^{3/4}\left|\langle\!\langle u_1 + x_{\varepsilon,n}\rangle\!\rangle\right| \le Cn^{5/2}\varepsilon^{1/4}\int_{\mathcal{V}} \widehat{\Phi}_n(\mathbf{x})\exp\{n\widehat{F}_n(\mathbf{x})\}\,d\mathbf{x},$$

with the same notations as in (5.17). We have the following expansions:

$$\begin{aligned} F_{n,1}(u_1,u_2) =& \mathcal{L}_n(x_{\varepsilon,n}^2) - (x_{\varepsilon,n}-\varepsilon)^2 - \kappa_1(u+x_{\varepsilon,n})^2 - \widetilde{\kappa}_2 u_2^2 + \\ & + O(|u_1 + x_{\varepsilon,n}|^3 + |u_2|^3); \\ F_{n,2}(s,t,r) =& -\mathcal{L}_n(x_{\varepsilon,n}^2) + (x_{\varepsilon,n}-\varepsilon)^2 - (\kappa_3 s^2 + \kappa_4\tau^2 + \kappa_5\tau^3 + \kappa_6\tau^4 + r^2) + \\ & + O(|s|^3 + |\tau|^5), \end{aligned}$$

where $t = ix_{\varepsilon,n} + (\pm 1 + i(1+\varepsilon - x_{\varepsilon,n}))\tau$, $\kappa_1,\kappa_2,\kappa_3,\kappa_4$ are defined in (3.7), (3.22), (3.20), (3.23) and κ_5,κ_6 satisfy (5.18). It is easy to see that $\kappa_1 \ge C\varepsilon/x_{\varepsilon,n}$,

$\kappa_3 \geq C$, $\Re\kappa_{4,5} > 0$, $\Re\kappa_6 \geq C > 0$. Thus

$$nF_{n,1}(u_1,u_2) \leq nF_{n,1}(x_{\varepsilon,n},0) - C\log^2(n\varepsilon/x_{\varepsilon,n}),$$
$$\max\{|u_1+x_{\varepsilon,n}|,|u_2|\} > (n\varepsilon/x_{\varepsilon,n})^{-1/2}\log(n\varepsilon/x_{\varepsilon,n}),$$
$$n\Re F_{n,2}(s,t,r) \leq nF_{n,2}(0,ix_{\varepsilon,n},0) - C\log^2 n, \quad \max\{|s|,|r|\} > n^{-1/2}\log n,$$
$$n\Re F_{n,2}(s,t,r) \leq nF_{n,2}(0,ix_{\varepsilon,n},0) - C\log^2 n, \quad |t-ix_{\varepsilon,n}| > n^{-1/4}\log^{1/2} n.$$

Since $n\varepsilon/x_{\varepsilon,n} \geq cn\varepsilon^{2/3} \geq cn^{1/3}$, the above bounds allow to restrict the integration to the neighbourhood. Make a change $u_1 = -x_{\varepsilon,n}+(n\varepsilon/x_{\varepsilon,n})^{-1/2}\widetilde{u}_1$, $u_2 = (n\varepsilon/x_{\varepsilon,n})^{-1/2}\widetilde{u}_2$, $t = ix_{\varepsilon,n}+n^{-1/4}(\pm1+i(1+\varepsilon-x_{\varepsilon,n}))\widetilde{t}_\pm$, $s = n^{-1/2}\widetilde{s}$, $r = n^{-1/2}\widetilde{r}$, then the multiplier before the integral is $C\dfrac{n^{5/2}\varepsilon^{1/4}}{n^{5/4}(n\varepsilon/x_{\varepsilon,n})}$. One can compute the derivatives of $\widetilde{\varphi}_n(u_1,u_2,s,t)$ and check that $\widetilde{\varphi}_n(-x_{\varepsilon,n},0,0,ix_{\varepsilon,n}) \leq C(\varepsilon/x_{\varepsilon,n})^2$, the derivatives of the first order at $(-x_{\varepsilon,n},0,0,ix_{\varepsilon,n})$ are $O(\varepsilon)$, the derivatives of the second order at $(-x_{\varepsilon,n},0,0,ix_{\varepsilon,n})$ are $O(\varepsilon/x_{\varepsilon,n})$ and the derivatives of the third order and higher at $(-x_{\varepsilon,n},0,0,ix_{\varepsilon,n})$ are $O(1)$. Thus

$$|\widetilde{\varphi}_n(u_1,u_2,s,t)| \leq C\Big(\varepsilon^2 x_{\varepsilon,n}^{-2} + \frac{\varepsilon}{n^{1/4}} + \frac{\varepsilon^{1/2}x_{\varepsilon,n}^{1/2}}{n^{1/2}} + \frac{\varepsilon}{n^{1/2}x_{\varepsilon,n}} + \frac{1}{n^{3/4}} + \frac{x_{\varepsilon,n}^{3/2}}{n^{3/2}\varepsilon^{3/2}}\Big)\times$$
$$\times \mathcal{P}(\widetilde{u},\widetilde{t}_\pm,\widetilde{s})$$

where $\mathcal{P}(\widetilde{u},\widetilde{t}_\pm,\widetilde{s})$ is a polynomial. Observe that $|u_1+x_{\varepsilon,n}| = \dfrac{|\widetilde{u}_1|}{(n\varepsilon/x_{\varepsilon,n})^{1/2}}$. Thus we can write

$$\varepsilon^{3/4}\left|\langle\langle u_1+x_{\varepsilon,n}\rangle\rangle\right| \leq \frac{Cn^{5/2}\varepsilon^{1/4}}{n^{5/4}(\frac{n\varepsilon}{x_{\varepsilon,n}})^{3/2}}\int_{\mathcal{V}}|\widetilde{u}_1|\cdot|\widetilde{\varphi}_n(u_1,u_2,t,s)|\exp\{n\,\widehat{F}_n(\widetilde{\mathbf{x}})\}\,d\mathbf{x} \leq$$
$$\leq \frac{C}{(n\varepsilon)^{1/4}}\cdot\Big(\varepsilon x_{\varepsilon,n}^{-1/2} + \frac{x_{\varepsilon,n}^{3/2}}{n^{1/4}} + \frac{x_{\varepsilon,n}^2}{(n\varepsilon)^{1/2}} + \frac{x_{\varepsilon,n}^{1/2}}{n^{1/2}} + \frac{x_{\varepsilon,n}^{3/2}}{(n\varepsilon)^{3/4}\varepsilon^{1/4}} + \frac{x_{\varepsilon,n}^3}{(n\varepsilon)^{3/2}\varepsilon}\Big),$$

which is bounded, since $(1+c)\varepsilon \leq x_{\varepsilon,n} \leq Cx_{\varepsilon,n}^{1/3}$ and $\varepsilon > \dfrac{a}{n}$.

$\square$

Chapter 6

Rate of convergence

In this chapter we provide a proof of Theorem 0.3. We split proof into three parts using the following inequality:

$$
\begin{aligned}
&\Big|\mathbb{E}_{H_n}\Big\{\frac{1}{n}\sum_{j=1}^{n} h(z_j)\Big\} - \int h(z)\rho(z)\,\mathbf{d}^2 z\Big| \leq \\
&\leq \Big|\mathbb{E}_{H_n}\Big\{\frac{1}{n}\sum_{j=1}^{n} h(z_j)\Big\} - \int h(z)\rho_{\varepsilon,n,\omega}(z)\,\mathbf{d}^2 z\Big| + \\
&+ \int |h(z)|\,|\rho_{\varepsilon,n,\omega}(z) - \widehat{\rho}_{\varepsilon,n,\omega}(z)|\,\mathbf{d}^2 z + \int |h(z)|\,|\widehat{\rho}_{\varepsilon,n,\omega}(z) - \rho_{\varepsilon,\omega}(z)|\,\mathbf{d}^2 z + \\
&+ \int |h(z)|\cdot|\rho_{\varepsilon,\omega}(z) - \widehat{\rho}_{\omega}(z)|\,\mathbf{d}^2 z
\end{aligned}
\tag{6.1}
$$

where $\widehat{\rho}_{\varepsilon,n,\omega}(z) = \dfrac{1}{\pi}\Big(\dfrac{|\mathrm{tr}_n(A_n - z)G^2(x^2_{\varepsilon,n})|^2}{\mathrm{tr}_n G^2(x^2_{\varepsilon,n}) + \varepsilon/(2x^3_{\varepsilon,n})} + x^2_{\varepsilon,n} \cdot \mathrm{tr}_n G(x^2_{\varepsilon,n})\widetilde{G}(x^2_{\varepsilon,n})\Big)$. Take $\varepsilon = n^{-2/5}$. In order to estimate the first summand, observe that $\lim\limits_{\varepsilon\to 0}\int h(z)\rho_{\varepsilon,n,\omega}(z)\,\mathbf{d}^2 z = \mathbb{E}_{H_n}\Big\{\frac{1}{n}\sum_{j=1}^{n} h(z_j)\Big\}$ uniformly in n according to Proposition 1.1, and Theorem 5.1 shows that

$$
\Big|\mathbb{E}_{H_n}\Big\{\frac{1}{n}\sum_{j=1}^{n} h(z_j)\Big\} - \int h(z)\rho_{\varepsilon,n,\omega}(z)\,\mathbf{d}^2 z\Big| \leq C\varepsilon^{1/2} = Cn^{-1/5}.
$$

To estimate $|\rho_{\varepsilon,n,\omega}(z) - \widehat{\rho}_{\varepsilon,n,\omega}(z)|$, we adjust the argument in Section 3.2 and Proposition 3.10 for $\varepsilon = n^{-2/5}$ instead of fixed ε. As in Section 3.2, we have

$$\mathbf{I}_{g_n} = Cn^3 \int_{\mathcal{V}} \Phi_n(\mathbf{x})\, e^{nF_n(\mathbf{x})}\, d\mathbf{x}, \text{ where}$$

$$\begin{aligned}
&\mathbf{x} = (u,s,t,r,v,w), \quad \mathcal{V} = [0,+\infty)\times\mathbb{R}\times L_t\times\mathcal{R}(t)\times\mathbb{R}\times[0,\sqrt{2u}],\\
&F_{n,1}(u) = \mathcal{L}_n(u^2) - (u-\varepsilon)^2,\ F_{n,2}(s,t,r) = -\Big(\mathcal{L}_n(s^2-t^2) + (t-i\varepsilon)^2 + r^2\Big),\\
&F_n(\mathbf{x}) = F_{n,1}(u) + F_{n,2}(s,t,r) - 2\varepsilon w^2 - \varepsilon v^2,\\
&\Phi_n(\mathbf{x}) = \frac{r-it-\varepsilon}{\sqrt{v^2+4r-4it-4\varepsilon}}\cdot\frac{u}{\sqrt{2u-w^2}}\cdot g_n(u^2, s^2-t^2).
\end{aligned}$$

We can change contours as in Section 3.2, since $x_{\varepsilon,n} \ge \kappa_0 > 0$. It is easy to check that

$$\begin{aligned}
&nF_{n,1}(u) \le nF_{n,1}(x_{\varepsilon,n}) - C\log^2 n, \quad |u-x_{\varepsilon,n}| > n^{-1/2}\log n,\\
&n\Re F_{n,2}(s,t,r) \le nF_{n,2}(0, ix_{\varepsilon,n}, 0) - C\log^2 n,\\
&\qquad\qquad\qquad\qquad \max\{|t-ix_{\varepsilon,n}|, |s|, |r|\} > n^{-1/2}\log n,\\
&-n\varepsilon v^2 - 2n\varepsilon w^2 \le C\log^2(n\varepsilon), \quad \max\{|v|,|w|\} > (n\varepsilon)^{-1/2}\log(n\varepsilon).
\end{aligned}$$

Since $n\varepsilon = n^{3/5}$, we can restrict the integration to a neighbourhood: $|u-x_{\varepsilon,n}|$, $|s|$, $|t-ix_{\varepsilon,n}|$, $|r| < n^{-1/2}\log n$, $|v| < (n\varepsilon)^{-1/2}\log(n\varepsilon)$, $0 \le w < (n\varepsilon)^{-1/2}\log(n\varepsilon)$. Make a change $u = x_{\varepsilon,n} + n^{-1/2}\widetilde{u}$, $s = n^{-1/2}\widetilde{s}$, $t = ix_{\varepsilon,n} + n^{-1/2}\widetilde{t}$, $r = n^{-1/2}\widetilde{r}$, $v = (n\varepsilon)^{-1/2}\widetilde{v}$, $w = (n\varepsilon)^{-1/2}\widetilde{w}$. Observe that

$$\begin{aligned}
\Phi_n(\widetilde{\mathbf{x}}) &= \alpha_{n,\varepsilon} g_n(x_{\varepsilon,n}^2, x_{\varepsilon,n}^2) + n^{-1/2}\mathcal{P}_1(\widetilde{u},\widetilde{t},\widetilde{r}) + n^{-1}\mathcal{P}_2(\widetilde{u},\widetilde{t},\widetilde{r},\widetilde{s}) +\\
&\quad + (n\varepsilon)^{-1}\mathcal{Q}_2(\widetilde{v},\widetilde{w}) + O((n\varepsilon)^{-3/2}\log^3(n\varepsilon)),\\
e^{nF_n(\widetilde{\mathbf{x}})} &= e^{-\kappa_1\widetilde{u}^2-\kappa_1\widetilde{t}^2-\kappa_3\widetilde{s}^2-\widetilde{r}^2-\varepsilon\widetilde{v}^2-2\varepsilon\widetilde{w}^2}\cdot\big(1 + n^{-1/2}\mathcal{P}_3(\widetilde{u},\widetilde{t},\widetilde{s}) + O(n^{-1}\log^4 n)\big),\\
\Phi_n(\widetilde{\mathbf{x}})\, e^{nF_n(\widetilde{\mathbf{x}})} &= \Big(\alpha_{n,\varepsilon} g_n(x_{\varepsilon,n}^2, x_{\varepsilon,n}^2) + n^{-1/2}(\mathcal{R}_1(\widetilde{\mathbf{x}}) + \mathcal{R}_3(\widetilde{\mathbf{x}})) +\\
&\quad + n^{-1}(\mathcal{R}_2(\widetilde{\mathbf{x}}) + \mathcal{R}_4(\widetilde{\mathbf{x}})) + (n\varepsilon)^{-1}\widehat{\mathcal{Q}}_2(\widetilde{v},\widetilde{w}) + O((n\varepsilon)^{-3/2}\log^3(n\varepsilon))\Big)\times\\
&\quad \times e^{-\kappa_1\widetilde{u}^2-\kappa_1\widetilde{t}^2-\kappa_3\widetilde{s}^2-\widetilde{r}^2-\varepsilon\widetilde{v}^2-2\varepsilon\widetilde{w}^2}
\end{aligned}$$

where $\mathcal{P}_{1,2,3}$, $\mathcal{R}_{1,2,3,4}$, $\mathcal{Q}_2$, $\widehat{\mathcal{Q}}_2$ are homogeneous polynomials with bounded coefficients, $\deg \mathcal{P}_k = k$, $\deg \mathcal{R}_k = k$, $\deg \mathcal{Q}_2 = \widehat{\mathcal{Q}}_2 = 2$. Taking into account that $\displaystyle\int u^{2k+1} e^{-cu^2}\, du = 0$, we obtain

$$\mathbf{I}_{g_n} = \frac{C}{\varepsilon} \int_{\mathcal{V}} \Phi_n(\widetilde{\mathbf{x}})\, e^{nF_n(\widetilde{\mathbf{x}})}\, d\widetilde{\mathbf{x}} = \frac{C}{\varepsilon}\Big(\beta_{\varepsilon,n} g_n(x^2_{\varepsilon,n}, x^2_{\varepsilon,n}) + O((n\varepsilon)^{-1})\Big) =$$
$$= C(\varepsilon, n, z) g_n(x^2_{\varepsilon,n}, x^2_{\varepsilon,n}) + O(n^{-1}\varepsilon^{-2}) = C(\varepsilon, n, z) g_n(x^2_{\varepsilon,n}, x^2_{\varepsilon,n}) + O(n^{-1/5})$$

Taking $g_n(\cdot) = \varphi_n(\cdot, z, z)$ and applying the same argument as in Section 3.2, we obtain

$$\mathbf{I}_{g_n} = \frac{g_n(x^2_{\varepsilon,n}, x^2_{\varepsilon,n})}{\varphi_n(x^2_{\varepsilon,n}, x^2_{\varepsilon,n}, z, z)} + O(n^{-1/5})$$

The last identity gives us (3.25) with $\alpha = 1/5$. We can also adjust the proof of Proposition 3.10 the same way and obtain (3.26) with $\alpha = 1/5$. Thus, in (3.31) we have

$$\rho_{\varepsilon,n,\omega}(z) = \frac{1}{\pi}\Big(\frac{|\mathrm{tr}_n(A_n - z)G^2(x^2_{\varepsilon,n})|^2}{\mathrm{tr}_n G^2(x^2_{\varepsilon,n}) + \varepsilon/(2x^3_{\varepsilon,n})} + x^2_{\varepsilon,n} \cdot \mathrm{tr}_n G(x^2_{\varepsilon,n}) \widetilde{G}(x^2_{\varepsilon,n})\Big) + O(n^{-1/5}) =$$
$$= \widehat{\rho}_{\varepsilon,n,\omega}(z) + O(n^{-1/5}),$$

which gives as an estimation of the second summand in (6.1).

Finally, one can use the relations (4.3) and (4.4), condition (C6) and Proposition 3.2 to obtain that $|\widehat{\rho}_{\varepsilon,n,\omega}(z) - \rho_{\varepsilon,\omega}(z)| \le Cn^{-1/5}$ and $|\rho_{\varepsilon,\omega}(z) - \widehat{\rho}_{\omega}(z)| \le C\varepsilon = Cn^{-2/5}$, which finishes the proof.

Conclusions

Evidently, if it is known that the limiting density of X_n can be found as a double limit

$$\rho(z) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{4\pi n} \Delta\, \mathbb{E}_{H_n} \Big\{ \log \det(Y(z) + \varepsilon^2) \Big\} \tag{6.2}$$

then supersymmetric integration method provides a plan of computing this double limit by deriving an integral representation of $\mathcal{Z}(\varepsilon, \varepsilon, z, z_1) = \mathbb{E}_{H_n} \left\{ \dfrac{\det(Y(z_1) + \varepsilon_1^2)}{\det(Y(z) + \varepsilon^2)} \right\}$. However, the verification of (6.2) turned out to be a rather complicated problem. It is a usual problem in non-hermitian case, and the most common way to resolve it is to obtain a bound on the smallest eigenvalue of X_n. We used another approach.

One can compare the result obtained in this paper with the result of [31]. The first major difference lies in the conditions on A_n. If one translate the conditions in [31] into the language of matrices, they roughly mean that the normalised counting measure of A_n should converge. In this paper we only need the normalised counting measure of $(A_n - z)(A_n - z)^*$ to converge in a 'nice' way, with some additional condition (C5) on convergence of non-hermitian components. The second major difference is that Free Probability Theory does not allow to estimate the rate of convergence, which is done in this paper in Section 6.

We got a polynomial estimation on the convergence of NCM of order $n^{-1/5}$. This bound does not seem to be the optimal one, since for such ensembles the right bound is usually of order n^{-1} or $n^{-1/2}$. However, there are some tricks, which allow to stricten the bound (for example, in [22, Section 11.3]).

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