Міністерство освіти і науки України
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роботу виконав:
студент групи М-162, 2 курсу
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Левченко М. А.
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Керівник: док. фіз.-мат. наук, проф.
Фаворов С. Ю.

Рецензент: канд. фіз.-мат. наук,
доцент кафедри вищої математики
Національного технічного
університету
«Харківський політехнічний інститут»
Гиря Н. П.

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1 Introduction

A function $\rho : M \times M \to \mathbb{R}^+$ is called a *metric* on the set M if it satisfies the following four axioms:

1. $\rho(x, x) = 0$ for every $x \in M$;
2. $\rho(x, y) = 0$ implies $x = y$ for all $x, y \in M$ (non-degeneracy);
3. $\rho(x, y) = \rho(y, x)$ for all $x, y \in M$ (symmetry);
4. $\rho(x, z) \leq \rho(x, y) + \rho(x, z)$ for all $x, y, z \in M$ (triangle inequality).

The pair (M, ρ) is then called a *metric space.*
A sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of the metric space M is called *fundamental* (or a *Cauchy sequence*) if for every $\varepsilon > 0$ there is a $N \in \mathbb{N}$ such that $\rho(x_n, x_m) < \varepsilon$ for all $n, m > N$. Every convergent sequence is fundamental, but the converse statement is not a priori true. Metric spaces for which it is true are called *complete.*
A set X together with defined on it operations of addition and multiplication by real scalars is called a real *vector space* (or *linear space*) if X is an abelian group with respect to addition, and for any $\lambda, \mu \in \mathbb{R}$ and any $x, y \in X$ the following relations hold:

- $1 \cdot x = x$;
- $(\lambda\mu)x = \lambda(\mu x)$;
- $(\lambda + \mu)x = \lambda x + \mu x$;
- $\lambda(x + y) = \lambda x + \lambda y$.

Let X be a real vector space. A map $|| \cdot ||$ from X to $\mathbb{R}^+$ is called a *norm* on X if it obeys the following axioms:

1. $||x|| = 0$ implies $x = 0$ for every $x \in X$ (non-degeneracy);
2. $||\lambda x|| = |\lambda| \cdot ||x||$ for every $x \in X$ and $\lambda \in \mathbb{R}$;
3. $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in X$ (triangle inequality).

The pair $(X, || \cdot ||)$ is then called a real *normed space.* Every normed space is also a metric space with the metric ρ defined by

$$\rho(x, y) = ||x - y||.$$

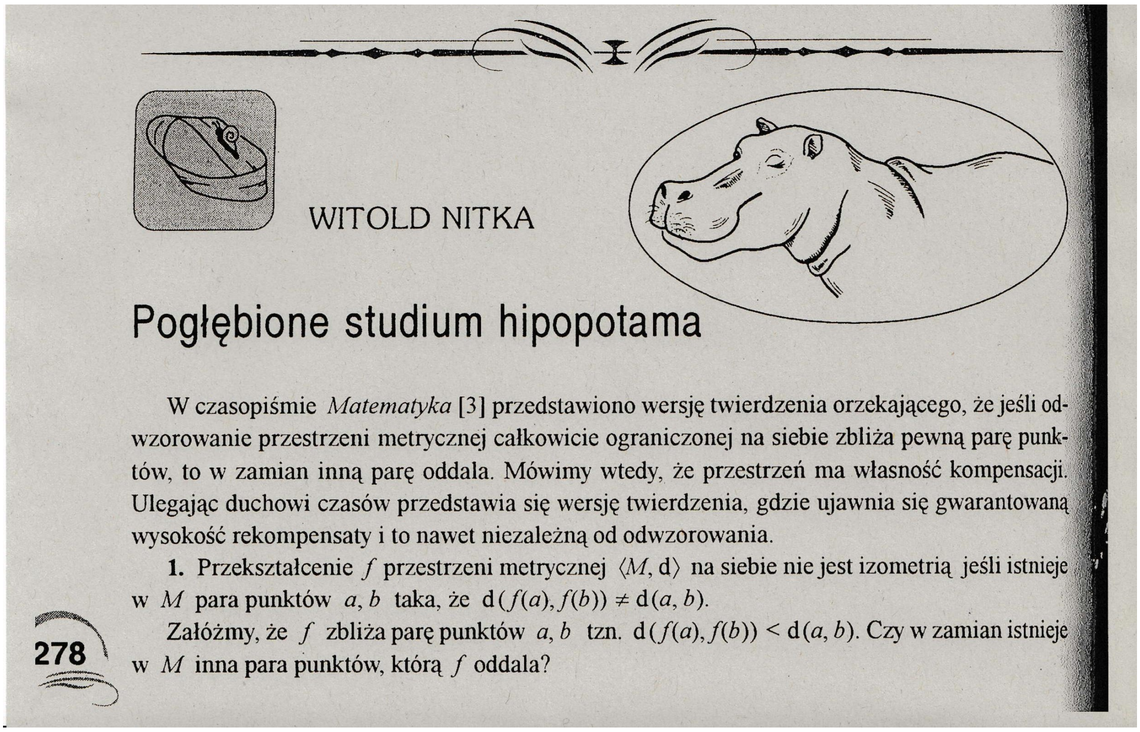
For a normed space X we denote its closed unit ball and the unit sphere by B_X and S_X respectively:

$$B_X = \{x \in X : ||x|| \leq 1\},$$

$$S_X = \{x \in X : ||x|| = 1\}.$$

A normed space is called a *Banach space* if it is complete in the metric induced by the norm. Let (M, ρ) be a metric space. A map $F : M \to M$ is called *non-expansive* if $\rho(F(x), F(y)) \leq \rho(x, y)$ for every $x, y \in M$; it is called *non-contractive* if $\rho(F(x), F(y)) \geq \rho(x, y)$ for every $x, y \in M$; $F : M \to M$ is called an *isometry* if it is both non-contractive and non-expansive, i.e. $\rho(F(x), F(y)) = \rho(x, y)$ for every $x, y \in M$.

A metric space (M, ρ) is said to be *plastic* if every non-expansive bijective map $F : M \to M$ is an isometry. Such spaces are called *Expand-Contract plastic* in [3]. A closely related concept is that of *Contract-Expand plastic* spaces, also known as *hippopotamus spaces* [2]: a metric space M is called Contract-Expand plastic if every non-expansive surjection from M onto itself is an isometry.



WITOLD NITKA

Pogłębione studium hipopotama

W czasopiśmie *Matematyka* [3] przedstawiono wersję twierdzenia orzekającego, że jeśli odwzorowanie przestrzeni metrycznej całkowicie ograniczonej na siebie zbliża pewną parę punktów, to w zamian inną parę oddala. Mówimy wtedy, że przestrzeń ma własność kompensacji. Ulegając duchowi czasów przedstawia się wersję twierdzenia, gdzie ujawnia się gwarantowaną wysokość rekompensaty i to nawet niezależną od odwzorowania.

1. Przekształcenie f przestrzeni metrycznej $\langle M, \mathrm{d} \rangle$ na siebie nie jest izometrią jeśli istnieje w M para punktów a, b taka, że $\mathrm{d}(f(a), f(b)) \neq \mathrm{d}(a, b)$.

Załóżmy, że f zbliża parę punktów a, b tzn. $\mathrm{d}(f(a), f(b)) < \mathrm{d}(a, b)$. Czy w zamian istnieje w M inna para punktów, którą f oddala?

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As demonstrated in [3], a totally bounded space is plastic, but the converse statement is not necessarily true, and there are even examples of unbounded plastic spaces. If X is an infinite-dimensional Banach space, then both its unit ball B_X and the unit sphere S_X are not totally bounded. The unit balls of some Banach spaces are shown to be plastic [4–10]. Whether B_X (or S_X) is plastic for every Banach space X remains an open question for the time being.

The following general theorem about non-expensive bijections on unit balls of Banach spaces was proven in [4].

Theorem 1.1. *Let X be a real Banach space, and let $F : B_X \to B_X$ be a non-expansive bijection. Then the following hold:*

1. $F(0) = 0$;
2. $F^{-1}(S_X) \subseteq S_X$;
3. *If $F(x)$ is an extreme point of B_X, then $F(ax) = aF(x)$ for every $a \in (0, 1)$;*
4. *If x is an extreme point of B_X, then $F^{-1}(x)$ is also an extreme point of B_X;*

5. *If x is an extreme point of B_X, then $F(-x) = -F(x)$.*

Moreover, if X is strictly convex, then

(i) F maps the unit sphere bijectively onto itself;

(ii) $F(ax) = aF(x)$ for all $x \in S_X$ and $a \in (0,1)$;

(iii) $F(-x) = -F(x)$ for all $x \in S_X$.

As the title suggests, this paper deals with the plasticity of the unit spheres of some (real) Banach spaces. In particular, we show that for a Hilbert space H the plasticity of its unit sphere S_H is a simple consequence of the parallelogram law. Then we demonstrate that the unit sphere of the sequence space ℓ_1 is plastic. In the last section some partial results about non-expansive bijections on the unit sphere of the sequence space ℓ_∞ are obtained.

For an element x of a Banach sequence space we denote its n-th coordinate by x^n, and by $\texttt{supp}(x)$ we denote the set of all such $k \in \mathbb{N}$ that $x^k \neq 0$. For every $k \in \mathbb{N}$ we denote by e_k the vector that has 1 as its k-th coordinate and 0 as its j-th coordinate for every $j \neq k$.

A subset A of a vector space is called *convex* if for every $x, y \in A$ it contains the line segment

$$[x,y] = \{tx + (1-t)y : t \in [0,1]\}.$$

An element of a convex set A is called an *extreme point* of A if it does not lie in the interior of any line segment contained in A.

For a convex set A we denote by $\texttt{ext}(A)$ the set of extreme points of A.

A Banach space X is called *strictly convex* if its unit sphere S_X contains no non-trivial line segments, i.e. $S_X = \texttt{ext}(B_X)$. Equivalently, X is strictly convex if and only if $||x - y|| = 2$ implies that $y = -x$ for every $x, y \in S_X$.

2 Hilbert spaces

Let X be a real vector space. A map $\langle\cdot,\cdot\rangle$ from $X \times X$ to $\mathbb{R}$ is called a real *inner product* (or a *scalar product*) if it obeys the following axioms:

1. $\langle x, x\rangle \geq 0$ for all $x \in X$ (positivity);
2. $\langle x, x\rangle = 0$ implies that $x = 0$ (non-degeneracy);
3. $\langle \lambda x_1 + \mu x_2, y\rangle = \lambda\langle x_1, y\rangle + \mu\langle x_2, y\rangle$ for all $x_1, x_2, y \in X$ and all $\lambda, \mu \in \mathbb{R}$ (linearity in the first argument);
4. $\langle x, y\rangle = \langle y, x\rangle$ (symmetry).

The space X is then called an *inner product space.* Note that the last two axioms together imply the linearity in the second argument.

An inner product space is also a normed space with the norm $||\cdot||$ defined by

$$||x|| = \sqrt{\langle x, x\rangle}.$$

An inner product space H that is complete with respect to the metric induced by the inner product is called a *Hilbert space.*

An important example of a Hilbert space is the space ℓ_2 of square-summable real sequences with the inner product defined by

$$\langle x, y\rangle = \sum_{k=1}^{\infty} x^k y^k, \quad x, y \in \ell_2.$$

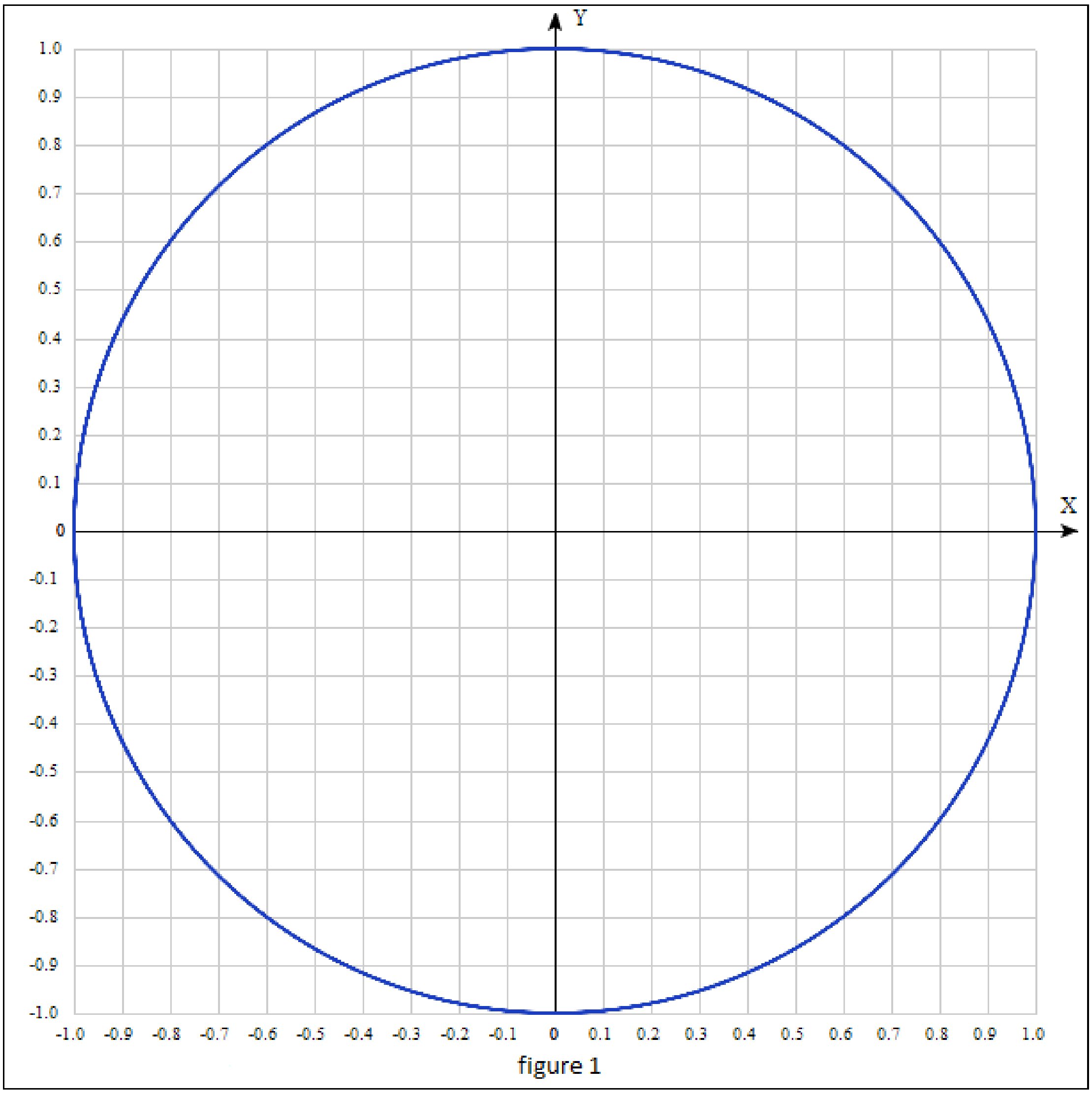


figure 1

In every inner product space H the following identity, known as the *parallelogram law*, holds for every $x, y \in H$:

$$||x+y||^2 + ||x-y||^2 = 2||x||^2 + 2||y||^2.$$

Lemma 2.1. *Let X be a strictly convex space, and let $F : S_X \to S_X$ be a non-contractive (or non-expansive) bijection. Then $F(-x) = -F(x)$ for every $x \in S_X$.*

Proof. Suppose first that $F : S_X \to S_X$ is a non-contractive bijection, and let $x \in S_X$. Then $||x - (-x)|| = 2 \leq ||F(x) - F(-x)|| \leq ||F(x)|| + ||F(-x)|| = 2$, so $||F(x) - F(-x)|| = 2 = ||F(x)|| + || - F(-x)||$, and the strict convexity of X implies that $F(x) = -F(-x)$.
Now suppose that $G : S_X \to S_X$ is a non-expansive bijection. Then G^{-1} is a non-contractive bijection, and since G^{-1} is surjective, there is a $y \in S_X$ such that $x = G^{-1}(y)$. Thus $G(-x) = G(-G^{-1}(y)) = G(G^{-1}(-y)) = -y = -G(x)$. □

Theorem 2.1. *Let H be a Hilbert space, and let $F : S_H \to S_H$ be a non-expansive bijection. Then F is an isometry.*

Proof. Let $x, y \in S_H$. Then

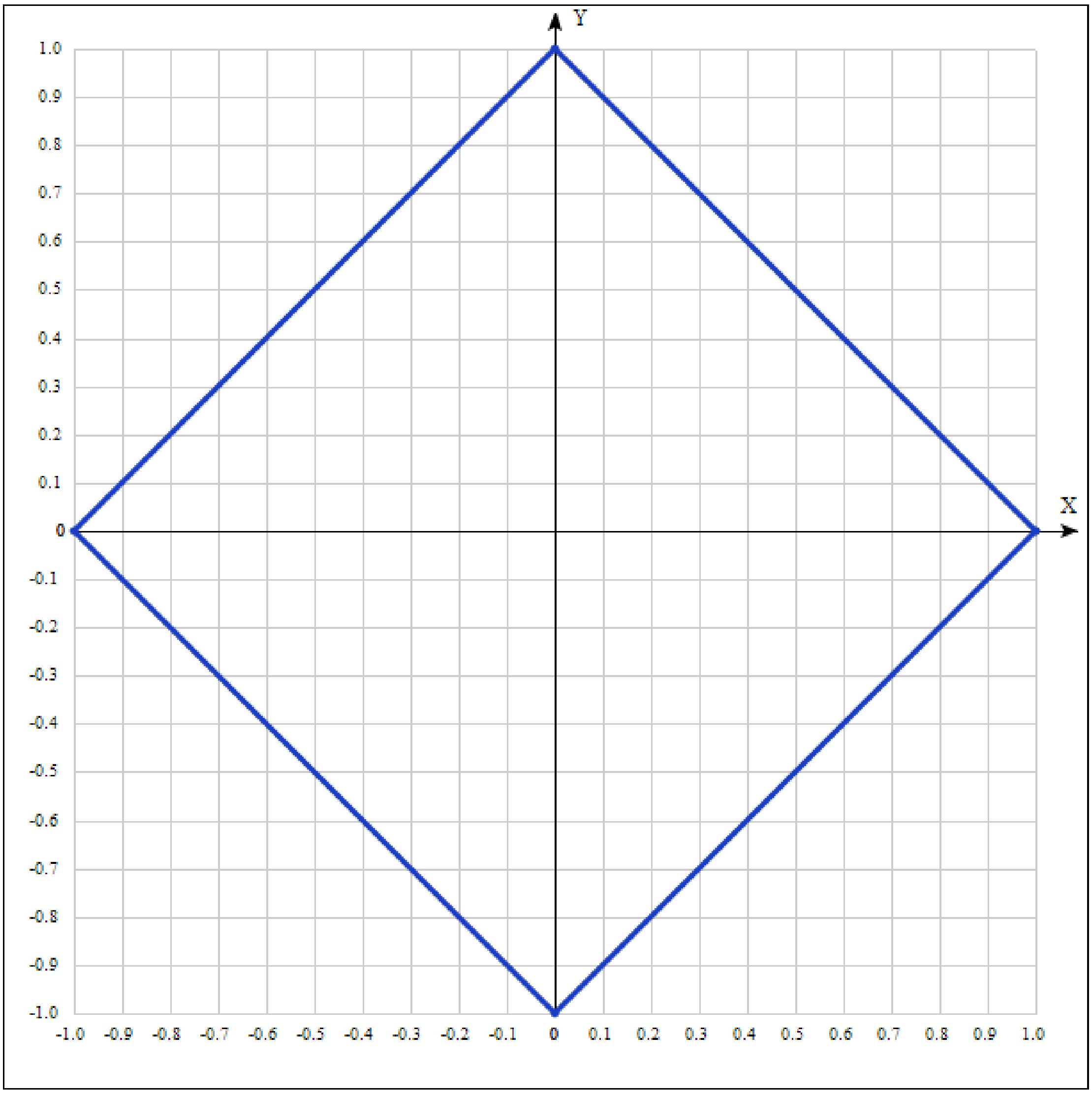
$$||x-y||^2 + ||x+y||^2 = ||F(x) - F(y)||^2 + ||F(x) + F(y)||^2 = 4.$$

The strict inequality $||x-y||^2 > ||F(x) - F(y)||^2$ thus implies $||x+y||^2 < ||F(x) + F(y)||^2$, a contradiction to the supposed non-expansiveness of F. $\square$

3 The space ℓ_1

The Banach sequence space ℓ_1 consists of all real sequences whose series are absolutely convergent with the norm defined by

$$||x|| = \sum_{k=1}^{\infty} |x^k|, \quad x \in \ell_1.$$



Lemma 3.1. *Let $p, q \in S_{\ell_1}$ be such that for every $x \in S_{\ell_1}$*

$$(||x - p|| = 2) \vee (||x - q|| = 2).$$

Then $p = \pm e_k$ for some $k \in \mathbb{N}$, and $q = -p$.

Proof. It is enough to show that the following two statements are true for such p, q:

$$\mathtt{supp}(p) = \mathtt{supp}(q), \tag{1}$$

and

$$|\mathtt{supp}(p)| = 1. \tag{2}$$

In contradiction to (1) suppose WLOG that $\mathtt{supp}(p) \nsubseteq \mathtt{supp}(q)$, i.e. there is a $k \in \mathbb{N}$ such that $p^k \neq 0$, $q^k = 0$, and let $x = \frac{1}{2}(q + \mathtt{sgn}(p^k)e_k)$. Then $||x - p|| < 2$, $||x - q|| < 2$.
Now in contradiction to (2) suppose that $|\mathtt{supp}(p)| \geq 2$, and let $k, j \in \mathtt{supp}(p) = \mathtt{supp}(q)$, $k \neq j$, $x = \frac{1}{2}(\mathtt{sgn}(p^k)e_k + \mathtt{sgn}(q^j)e_j)$. Then $||x - p|| < 2$, $||x - q|| < 2$. □

Lemma 3.2. *Let* $F : S_{\ell_1} \to S_{\ell_1}$ *be a non-expansive bijection. Then for every* $x \in \mathtt{ext}(B_{\ell_1})$ $F^{-1}(x) \in \mathtt{ext}(B_{\ell_1})$, *and* $F^{-1}(-x) = -F^{-1}(x)$.

Proof. Recall that $\mathtt{ext}(B_{\ell_1}) = \{\pm e_k\}_{k=1}^{\infty}$, and let $x \in \mathtt{ext}(B_{\ell_1})$.
Then $(||y - x|| = 2) \vee (||y + x|| = 2)$ for every $y \in S_{\ell_1}$, and thus $||F^{-1}(y) - F^{-1}(x)|| = 2$ or $||F^{-1}(y) - F^{-1}(-x)|| = 2$ for every $y \in S_{\ell_1}$. Since F^{-1} is bijective, the last statement is equivalent to $(||y - F^{-1}(x)|| = 2) \vee (||y - F^{-1}(-x)|| = 2)$ holding for all $y \in S_{\ell_1}$. Applying Lemma 3.1 completes the proof. □

Lemma 3.3. *Let* $F : S_{\ell_1} \to S_{\ell_1}$ *be a non-expansive bijection,* $N \in \mathbb{N}$. *Then*

$$F\left(\sum_{k=1}^{N} a_k F^{-1}(e_k)\right) = \sum_{k=1}^{N} a_k e_k \tag{3}$$

for every collection $\{a_k\}_{k=1}^{N}$ *of non-negative real scalars such that* $\sum_{k=1}^{N} a_k = 1$.

Proof. The case $N = 1$ is trivial. Now assume that the lemma holds for some $N - 1 \in \mathbb{N}$. We will show that the lemma holds for N, thus proving it by induction.
For every $n \in N$ denote $F^{-1}(e_n)$ by g_n, and fix some $x = \sum_{k=1}^{N} a_k g_k \in S_{\ell_1}$ such that $a_N \neq 0$, $a_k \geq 0$ for every $k \leq N$. Denote $(1 - a_N)^{-1} \sum_{k=1}^{N-1} a_k g_k \in S_{\ell_1}$ by $\tilde{x}$, so that $x = (1 - a_N)\tilde{x} + a_N g_N$. Then

$$||x - g_k|| = 1 - a_k + \sum_{j \neq k} a_j = 2(1 - a_k)$$

for every $k \leq N$;

$$||x - \tilde{x}|| = a_N + \sum_{k=1}^{N-1} \left| a_k - \frac{a_k}{1 - a_N} \right| = a_N + \left(\frac{1}{1 - a_N} - 1 \right) \sum_{k=1}^{N-1} a_k = a_N + 1 - 1 + a_N = 2a_N;$$

Now let $y = F(x)$, so $||y - e_N|| \leq 2(1 - a_N)$, $||y - F(\tilde{x})|| \leq 2a_N$. Then $y^N \geq 0$, and

$$||y - e_N|| = |y^N - 1| + \sum_{k=1}^{N-1} |y^k| = 1 - y^N + 1 - y^N = 2(1 - y^N) \leq 2(1 - a_N),$$

so $y^N \geq a_N$. Then

$$||y - F(\tilde{x})|| = \sum_{k=1}^{N-1} \left| y^k - \frac{a_k}{1 - a_N} \right| + \sum_{k=N}^{\infty} |y^k| \leq 2a_N \leq 2 \sum_{k=N}^{\infty} |y^k| \Rightarrow$$

$$\sum_{k=1}^{N-1}\left|y^k-\frac{a_k}{1-a_N}\right|\leq\sum_{k=N}^{\infty}|y^k|=1-\sum_{k=1}^{N-1}|y^k|\Rightarrow$$

$$\sum_{k=1}^{N-1}\left(\left|\frac{a_k}{1-a_N}-y^k\right|+|y^k|\right)\leq 1.$$

On the other hand,

$$\sum_{k=1}^{N-1}\left(\left|\frac{a_k}{1-a_N}-y^k\right|+|y^k|\right)\geq\sum_{k=1}^{N-1}\left|\frac{a_k}{1-a_N}-y^k+y^k\right|=(1-a_N)^{-1}\sum_{k=1}^{N-1}|a_k|=1,$$

so

$$\sum_{k=1}^{N-1}\left(\left|\frac{a_k}{1-a_N}-y^k\right|+|y^k|\right)=1. \tag{4}$$

The inequality $y^N\geq a_N$ means that $\sum_{k=1}^{N-1}|y^k|\leq 1-a_N$, together with (4) it implies that $\sum_{k=1}^{N-1}|\frac{a_k}{1-a_N}-y^k|\geq a_N$. Thus

$$2a_N\geq||F(\tilde{x})-y||\geq y^N+\sum_{k=1}^{N-1}\left|\frac{a_k}{1-a_n}-y^k\right|\geq 2a_N,$$

so all the inequalities in the chain above are in fact equalities, $y^N=\sum_{k=1}^{N-1}|\frac{a_k}{1-a_N}-y^k|=a_N$, $\sum_{k=1}^{N-1}|y^k|=1-a_N$, and thus $y=a_Ne_N+\sum_{k=1}^{N-1}y^ke_k$.
Now let A be the set of all such $k\leq N-1$ that $a_k\neq 0$. Then for every $k\in A$

$$||y-e_k||=1-y^k+\sum_{j\neq k}|y^j|=1-y^k+1-|y^k|=2(1-y^k)\leq 2(1-a_k),$$

so $y^k\geq a_k$ for all $k\in A$. Thus

$$1-a_N=\sum_{k=1}^{N-1}a_k=\sum_{k=1}^{N-1}|y^k|\geq\sum_{k\in A}y^k\geq\sum_{k\in A}a_k=1-a_N,$$

so $\sum_{k\in A}y^k=\sum_{k\in A}a_k=1-a_N$, and $y=\sum_{k=1}^{N}a_ke_k$. □

Lemma 3.4. *Let* $F:S_{\ell_1}\to S_{\ell_1}$ *be a non-expansive bijection,* $N\in\mathbb{N}$.
Then (3) holds for every collection $\{a_k\}_{k=1}^N$ *of real scalars such that* $\sum_{k=1}^N|a_k|=1$.

Proof. Fix some $\{a_k\}_{k=1}^N\in\mathbb{R}^N$ such that $\sum_{k=1}^N|a_k|=1$. Then

$$F\left(\sum_{k=1}^{N}a_kF^{-1}(e_k)\right)=F\left(\sum_{k=1}^{N}|a_k|F^{-1}(\mathtt{sgn}(a_k)e_k)\right).$$

Let $T : \ell_1 \to \ell_1$ be a linear isometry defined on the canonical basis by

$$Te_k = \begin{cases} \mathtt{sgn}(a_k)e_k, & k \le N, \\ e_k, & k > N, \end{cases}$$

and denote the restriction of T to S_{ℓ_1} by $\tilde{T}$.
Note that $T^{-1} = T$, and let $G = \tilde{T}F\tilde{T}$, so that $F = \tilde{T}G\tilde{T}$. Then

$$F\left(\sum_{k=1}^{N} |a_k| F^{-1}(\mathtt{sgn}(a_k)e_k)\right) = \tilde{T}G\tilde{T}\left(\sum_{k=1}^{N} |a_k| \tilde{T}G^{-1}\tilde{T}(\mathtt{sgn}(a_k)e_k)\right) =$$

$$= \tilde{T}G\tilde{T}\left(\sum_{k=1}^{N} |a_k| \tilde{T}G^{-1}(e_k)\right) = \tilde{T}G\tilde{T}^2\left(\sum_{k=1}^{N} |a_k| G^{-1}(e_k)\right) =$$

$$= \tilde{T}G\left(\sum_{k=1}^{N} |a_k| G^{-1}(e_k)\right) = \tilde{T}\left(\sum_{k=1}^{N} |a_k| e_k\right) = \sum_{k=1}^{N} |a_k| \mathtt{sgn}(a_k)e_k = \sum_{k=1}^{N} a_k e_k.$$

□

Corollary. Let $F : S_{\ell_1} \to S_{\ell_1}$ be a non-expansive bijection. Then $\{F^{-1}(e_k)\}_{k\in\mathbb{N}}$ is a Schauder basis in ℓ_1.

For every non-expansive bijection $F : S_{\ell_1} \to S_{\ell_1}$ let $T_F : \ell_1 \to \ell_1$ be the linear isometry defined on the canonical basis by

$$T_F e_k = F^{-1}(e_k), \quad k \in \mathbb{N},$$

and let $I_F = F\tilde{T}_F$, where $\tilde{T}_F$ is the restriction of T_F to S_{ℓ_1}. Then $I_F : S_{\ell_1} \to S_{\ell_1}$ is a non-expansive bijection, and $I_F(x) = x$ for every $x \in \mathtt{ext}(B_{\ell_1})$.

Lemma 3.5. *Let $F : S_{\ell_1} \to S_{\ell_1}$ be a non-expansive bijection. Then $I_F(x) = x$ for every $x \in S_{\ell_1}$.*

Proof. Let $x \in S_{\ell_1}$, $y = I_F(x)$. Then

$$|x^k| = \max\left(1 - \frac{||x - e_k||}{2}, 1 - \frac{||x + e_k||}{2}\right) = 1 - \frac{||x - e_k||}{2} + 1 - \frac{||x + e_k||}{2},$$

so

$$1 = ||x|| = \sum_{p\in\mathtt{ext}(B_{\ell_1})} \left(1 - \frac{||x - p||}{2}\right) = \sum_{p\in\mathtt{ext}(B_{\ell_1})} \left(1 - \frac{||y - p||}{2}\right) = ||y||.$$

The existence of such $p \in \mathtt{ext}(B_{\ell_1})$ that $||I_F(x) - I_F(p)|| = ||y - p|| < ||x - p||$ would imply the existence of such $p' \in \mathtt{ext}(B_{\ell_1})$ that $||y - p'|| = ||I_F(x) - I_F(p')|| > ||x - p'||$, contradicting the non-expansiveness of I_F. Thus $||x - p|| = ||y - p||$ for every $p \in \mathtt{ext}(B_{\ell_1})$, so $x^k = \frac{1}{2}(||x + e_k|| - ||x - e_k||) = \frac{1}{2}(||y + e_k|| - ||y - e_k||) = y^k$ for every $k \in \mathbb{N}$. □

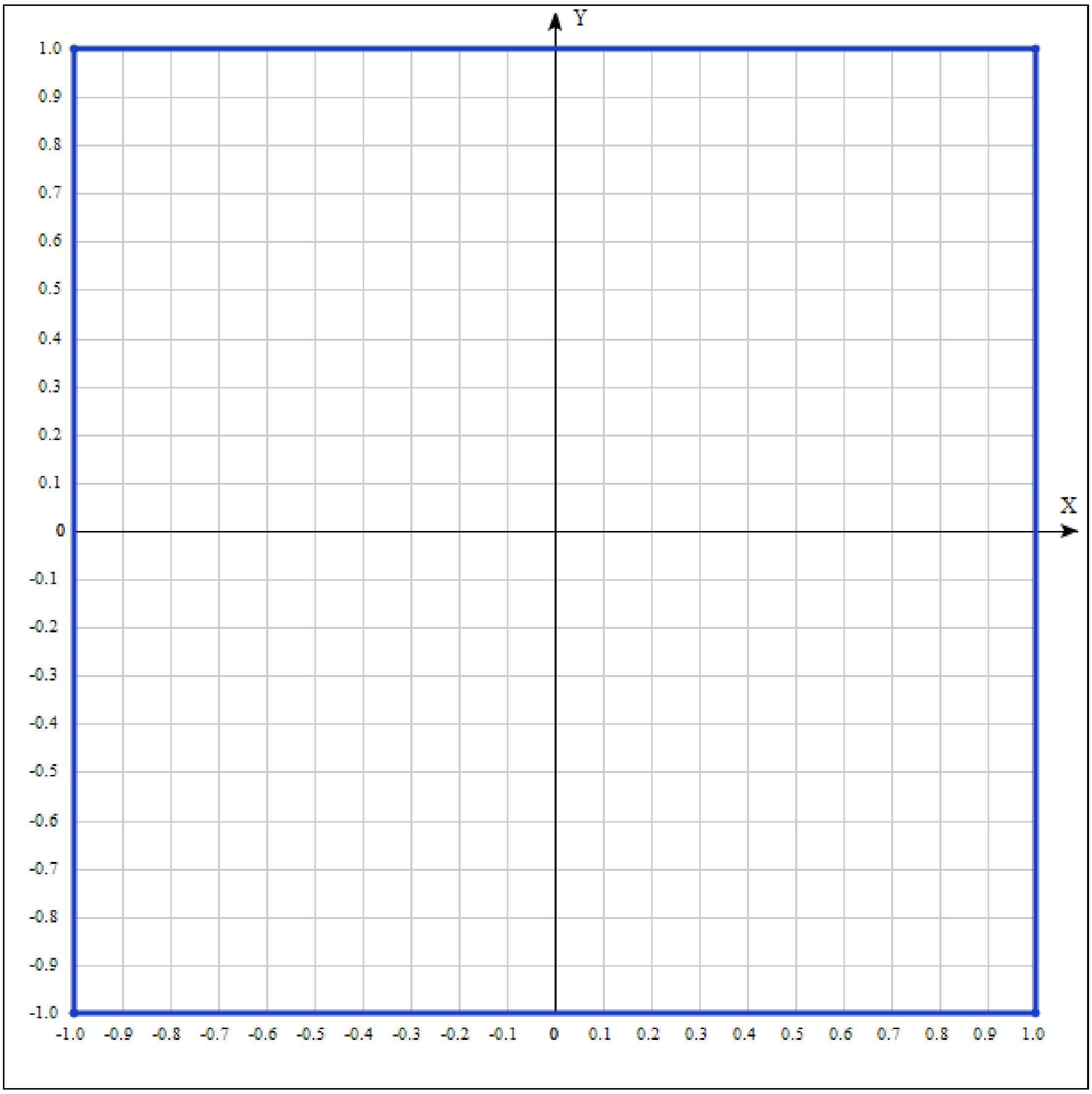
Theorem 3.1. *Let $F : S_{\ell_1} \to S_{\ell_1}$ be a non-expansive bijection. Then F is an isometry.*

Proof. Lemma 3.5 states that the composition of F and $\tilde{T}_F$ is the identity map on S_{ℓ_1}. Since $\tilde{T}_F$ is an isometry, F has to be an isometry as well. □

4 The space ℓ_∞

The Banach sequence space ℓ_∞ consists of all bounded real sequences with the norm defined by

$$||x|| = \sup_{k \in \mathbb{N}} |x^k|, \quad x \in \ell_\infty.$$



Lemma 4.1. *Let $p, q \in S_{\ell_\infty}$ be such that for every $x \in S_{\ell_\infty}$*

$$(||x - p|| = 2) \vee (||x - q|| = 2).$$

Then $p \in \texttt{ext}(B_{\ell_\infty})$, and $q = -p$.

Proof. Recall that $\texttt{ext}(B_{\ell_\infty}) = \{x \in S_{\ell_\infty} : |x^k| = 1,\ k \in \mathbb{N}\}$. If there is a $k \in \mathbb{N}$ such that $|p^k| < 1$, then $||e_k - p|| < 2$, $|| - e_k - p|| < 2$, so $||e_k - q|| = || - e_k - q|| = 2$, and that is impossible in S_{ℓ_∞}. Thus $|p^k| = |q^k| = 1$ for every $k \in \mathbb{N}$, and there is no $j \in \mathbb{N}$ such that $p^j = q^j$, since in the opposite case both distances $||p^j e_j - p||$ and $||p^j e_j - q||$ would have been equal to 1. $\square$

Corollary. Let $F : S_{\ell_\infty} \to S_{\ell_\infty}$ be a non-expansive bijection, $x \in \mathtt{ext}(B_{\ell_\infty})$.
Then $F^{-1}(x) \in \mathtt{ext}(B_{\ell_\infty})$, $F^{-1}(-x) = -F^{-1}(x)$.

Lemma 4.2. *Let* $x \in S_{\ell_\infty}$ *be such that* $||x-p|| \in (0,1)$ *for some* $p \in \mathtt{ext}(B_{\ell_\infty})$. *Then there is a* $q \in \mathtt{ext}(B_{\ell_\infty})$ *such that* $||x-q|| \in (1,2)$.

Proof. Let $A = \{k \in \mathbb{N} : |x^k - p^k| > \varepsilon\}$ for some $\varepsilon \in (0, ||x-p||)$. Then the following inequalities hold:

$$\sup_{k \in \mathbb{N}\setminus A} |x^k - p^k| \le \varepsilon, \quad 2 - \varepsilon \ge \sup_{k \in A} |x^k + p^k| \ge 1 + ||x-p||.$$

Now let $q \in \mathtt{ext}(B_{\ell_\infty})$ be defined by

$$q^k = \begin{cases} -p^k, & k \in A, \\ p^k, & k \in \mathbb{N} \setminus A. \end{cases}$$

Then $||x-q|| = \sup_{k\in\mathbb{N}} |x^k - q^k| = \max\left(\sup_{k\in A} |x^k + p^k|, \sup_{k\in\mathbb{N}\setminus A} |x^k - p^k|\right) = \sup_{k\in A}|x^k + p^k|$. ☐

Let E denote the set of all such vectors $x \in S_{\ell_\infty}$ that for every $p \in \mathtt{ext}(B_{\ell_\infty})$

$$(||x-p|| = 2) \Rightarrow (||x+p|| = 1). \tag{5}$$

Lemma 4.3. $E = \{\pm e_k\}_{k\in\mathbb{N}}$.

Proof. For an x of the form $\pm e_k$ (5) is obviously true, so it remains to prove that $E \subset \{\pm e_k\}_{k\in\mathbb{N}}$. Suppose in contradiction that $|\mathtt{supp}(x)| \ge 2$ for some $x \in E$. Then there is a $j \in \mathbb{N}$ such that $x^j \ne 0$, $||x - x^j e_j|| = 1$. Now let $p \in \mathtt{ext}(B_{\ell_\infty})$ be such that $||x - x^j e_j - p|| = 2$, and let $p' = p + (\mathtt{sgn}(x^j) - p^j)e_j$. Then $||x - p'|| = ||x-p|| = 2$, $||x + p'|| \ge |x^j + \mathtt{sgn}(x^j)| > 1$, contradicting (5). ☐

Lemma 4.4. *Let* $F : S_{\ell_\infty} \to S_{\ell_\infty}$ *be a non-expansive bijection,* $x \in E$. *Then* $F(x) \in E$.

Proof. Suppose in contradiction that $F(x) \notin E$, i.e. there exists such $p \in \mathtt{ext}(B_{\ell_\infty})$ that $||F(x) - p|| = 2$, $||F(x) + p|| \ne 1$. Lemma 4.2 implies that we may WLOG assume that $||F(x) + p|| > 1$. Then $||x - F^{-1}(p)|| = 2$, $||x - F^{-1}(-p)|| = ||x + F^{-1}(p)|| > 1$. Since $F^{-1}(p) \in \mathtt{ext}(B_{\ell_\infty})$, the last statement contradicts (5). ☐

For every $a \in \mathtt{ext}(B_{\ell_\infty})$ let $T_a : S_{\ell_\infty} \to S_{\ell_\infty}$ be the isometry of pointwise multiplication by a:

$$T_a(x^k) = (a^k x^k), \quad (x^k) \in S_{\ell_\infty}.$$

Now for every non-expansive bijection $F : S_{\ell_\infty} \to S_{\ell_\infty}$ let $F_1 = T_{F^{-1}(e)}F$, where $e = (1,1,1,...)$. The map $F_1 : S_{\ell_\infty} \to S_{\ell_\infty}$ is then a non-expansive bijection, $F_1(e) = e$, $F_1(-e) = -e$.
For every $k \in \mathbb{N}$ let $h_k = e - 2e_k$, and denote the set $\{h_k\}_{k\in\mathbb{N}}$ by H.

Lemma 4.5. *Let* $F : S_{\ell_\infty} \to S_{\ell_\infty}$ *be a non-expansive bijection,* $x \in H$. *Then* $F_1^{-1}(x) \in H$.

Proof. Denote by W the set of all vectors $y \in S_{\ell_\infty}$ such that

$$|\{k \in \mathbb{N} : y^k \leq 0\}| = |\{k \in \mathbb{N} : ||y + e_k|| \leq 1\}| \leq 1.$$

Then $W \cap \mathtt{ext}(B_{\ell_\infty}) = \{e\} \cup H$, so it is enough to show that $F_1^{-1}(x) \in W$. Now assume the contrary, i.e. there are $i, j \in \mathbb{N}$ such that $i \neq j$, $||F_1^{-1}(x) + e_i|| \leq 1$, $||F_1^{-1}(x) + e_j|| \leq 1$. The fact that $-e$ is a fixed point of F_1 implies that $F_1(-e_i) = -e_k$, $F_1(-e_j) = -e_l$ for some $k, l \in \mathbb{N}$. Then $||x + e_k|| \leq 1$, $||x + e_l|| \leq 1$, contradicting the assumption that $x \in H$. ☐

Lemma 4.6. *Let $F : S_{\ell_\infty} \to S_{\ell_\infty}$ be a non-expansive bijection. Then F_1 maps the set $\{e_k\}_{k\in\mathbb{N}}$ bijectively onto itself.*

Proof. Note that there is an injective map $\sigma : \mathbb{N} \to \mathbb{N}$ such that $F_1(e_k) = e_{\sigma(k)}$ for every $k \in \mathbb{N}$. We will show that σ is in fact a bijection. Now fix some $k \in \mathbb{N}$. Then $||e_{\sigma(k)} - h_{\sigma(k)}|| = 2$, and so $||F_1^{-1}(e_{\sigma(k)}) - F_1^{-1}(h_{\sigma(k)})|| = ||e_k - F_1^{-1}(h_{\sigma(k)})|| = 2$. Lemma 3.5 states that $F_1^{-1}(h_{\sigma(k)}) = h_j$ for some $j \in \mathbb{N}$. Note that $||e_k - h_j|| = 1$ for all $j \neq k$, so $j = k$. Thus $F_1^{-1}(\{h_{\sigma(k)}\}_{k\in\mathbb{N}}) = H$, so $\sigma(\mathbb{N}) = \mathbb{N}$. ☐

Theorem 4.1. *Let $F : S_{\ell_\infty} \to S_{\ell_\infty}$ be a non-expansive bijection. Then F maps $\mathtt{ext}(B_{\ell_\infty})$ bijectively onto itself.*

Proof. Lemma 4.6 states that there is a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $F_1(e_k) = e_{\sigma(k)}$ for every $k \in \mathbb{N}$. Define the bijective isometry $P : S_{\ell_\infty} \to S_{\ell_\infty}$ by

$$P(x^k) = (x^{\sigma^{-1}(k)}), \quad (x^k) \in S_{\ell_\infty},$$

and let $I = PF_1$. Then I is a non-expansive bijection, and $I(x) = x$ for every $x \in E$. Now fix an $x \in \mathtt{ext}(B_{\ell_\infty})$, and let $y = I^{-1}(x)$. Then $y \in \mathtt{ext}(B_{\ell_\infty})$, and the non-contractiveness of I^{-1} implies that $y = x$. Indeed, suppose in contradition that there is a $k \in \mathbb{N}$ such that $y^k = -x^k$. Then $||I^{-1}(x) - I^{-1}(y^k e_k)|| = ||y - y^k e_k|| = 1 < 2 = ||x - y^k e_k||$. Thus $I^{-1}(x) = x = I(x)$ for every $x \in \mathtt{ext}(B_{\ell_\infty})$. Now recall that $I = PF_1 = PT_{F^{-1}(e)}F$. Both P and $T_{F^{-1}(e)}$ map $\mathtt{ext}(B_{\ell_\infty})$ bijectively onto itself, and so does F. ☐

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