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на тему ***«Одометричні фактори несингулярних дій рангу один»***

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Анотації

Вєпрік М. І. Одометричні фактори несингулярних дій рангу один

Для довільної зліченної дискретної нескінченної групи G введено несингулярні дії рангу один. Для вкладеної послідовності $\Gamma_1 \supsetneqq \Gamma_2 \supsetneqq \cdots$ кофінітних підгруп у G з $\bigcap_{n=1}^{\infty} \bigcap_{g \in G} g\Gamma_n g^{-1} = \{1_G\}$, проективна границя однорідних G-просторів G/Γ_n при $n \to \infty$ є G-простором. Наділивши цей G-простір ергодичною несингулярною неатомарною мірою, ми отримаємо динамічну систему, яку називають несингулярним одометром. Знайдено необхідну і достатню умову того, що задана несингулярна система рангу один буде ізоморфною заданому несингулярному одометру. Також знайдена необхідна та достатня умова того, що заданий одометр є фактором заданої несингулярної системи рангу один.

Vieprik M. I. Odometer factors of nonsingular actions of rank one

For an arbitrary countable discrete infinite group G, nonsingular rank-one actions are introduced. Given a nested sequence $\Gamma_1 \supsetneqq \Gamma_2 \supsetneqq \cdots$ of cofinite subgroups in G with $\bigcap_{n=1}^{\infty} \bigcap_{g \in G} g\Gamma_n g^{-1} = \{1_G\}$, the projective limit of the homogeneous G-spaces G/Γ_n as $n \to \infty$ is a G-space. Endowing this G-space with an ergodic nonsingular nonatomic measure we obtain a dynamical system which is called a nonsingular odometer. A necessary and sufficient condition has been found that a given nonsingular rank-one system will be isomorphic to a given nonsingular odometer. We also found necessary

and sufficient condition for nonsingular rank-one system to factor onto an odometer.

Зміст

Вступ

One of the most well-known Dynamical Systems problems is the isomorphism problem: how to determine whether two arbitrary dynamical systems are isomorphic. If we consider the corresponding isomorphism equivalence relation, then it will not be Borel on the class of ergodic dynamical systems. But the restriction of this equivalence relation to the subclass of tranformations of rank one will be Borel. Thus, the problem of isomorphism for transformations of rank one is considered more promising to find a solution. The rank one transformatons form a wide class of dynamical systems. They appear as a source of examples and counterexamples in ergodic theory. A generic ergodic transformation is of rank one. In this paper, a partial case of the isomorphism problem is solved, namely, when one of the transformations is of rank one, and the other is an odometer. In previous works, a similar problem was considered for the action of the group $\mathbb{Z}$ ([11]) and the action of the group $\mathbb{Z}^d$ ([12]), where d is natural. In this work we deals with the actions of arbitrary countable infinite groups. Also, arbitrary nonsingular actions of groups are considered here, whereas in the previous works only measure-preserving ones were considered. In these general settings the problems under considerations are more difficult. We encounter new phenomena that were absent in the previous works.

We now briefly outline the content of this work, which consists of 5 sections. Section 1 is divided into 5 subsections. In §1.1 we define, for an arbitrary countable group G, nonsingular G-actions of rank one. According to this definition, a nonsingular G-action T is of rank one if T is free and T admits a refining sequence of Rokhlin towers that approximate both the entire σ-algebra of Borel subsets and the G-orbits and, in addition, the Radon-Nikodym derivative of T

is constant on each transposition of the levels within every tower (see Definition 1.1). Definition 1.1 can be considered as an *abstract* definition of rank one. In the case of probability preserving $\mathbb{Z}$-actions, there exist several equivalent *constructive* definitions of this concept [10]. One of the most useful of these is the *cutting-and-stacking construction*, which explicitly associates a rank-one transformation to a sequence of integer-valued parameters.[1] This transformation is defined on the unit interval. It preserves the Lebesgue measure. A natural generalization of this construction for general countable groups was suggested in [9] and [3] in similar but non-equivalent versions. We call it the (C, F)-construction. The most general version of the (C, F)-construction, including the versions from [9] and [3] as particular cases, appeared in [5]. However, [5] deals only with *measure preserving* actions. In §1.2–1.3 here we define *nonsingular* (C, F)-actions. §1.2 is preliminary: we define (C, F)-equivalence relations and related quasi-invariant (C, F)-measures. The nonsingular (C, F)-actions related to the (C, F)-equivalence relations and (C, F)-measures appear in §1.3. They include all the nonsingular rank-one transformations (and actions of Abelian groups) that have been studied earlier in the literature: see [1], [2], [13], [3], [4], [7] and references therein.

Important concept of telescoping for the parameters of (C, F)-actions is introduced in §1.4.

In §1.5 we discuss the case of nonsingular $\mathbb{Z}$-actions of rank one along intervals in more detail. It is shown in §1.5 that the (C, F)-construction in this case is equivalent to the classical cutting-and-stacking with a single tower at every step of the construction. However, in contrast with the measure preserving case, the towers are now divided into subtowers of different width.

[1] Thus, the class of rank-one transformations is parametrized with a nice Polish space of integer parameters. However, different sequences of parameters can define isomorphic rank-one maps. One of the most challenging open problems in this field is to find necessary and sufficient conditions on the parameters which determine isomorphic transformations.

Section 2 is devoted to description of finite factors of rank-one nonsingular actions. We remind that a *factor* of a dynamical system is an invariant sub-σ-algebra of measurable subsets. Equivalently, a factor of a system is a dynamical system which appears as the image of the original system under a nonsingular equivariant mapping. Hence, the finite factors of an ergodic G-action correspond to the G-equivariant mappings onto homegeneous G-spaces G/Γ, where Γ is a cofinite subgroup in G. Each nonsingular (C,F)-action is parametrized by an underlying sequence $\mathcal{T}=(C_n,F_{n-1},\kappa_n,\nu_{n-1})_{n=1}^{\infty}$ of (C,F)-parameters, where C_n and F_{n-1} are finite subsets of G, κ_n is a probability on C_n and ν_{n-1} is measure on F_{n-1} for each $n\in\mathbb{N}$. These parameters have to satisfy some conditions listed in §1.3. The following is the main result of §2 (a stronger version of it is proved in Theorem 2.3; see also Remark 2.4).

Theorem. *A nonsingular (C,F)-action action T of G has a finite factor G/Γ if and only there is a telescoping $\mathcal{T}=(C_n,F_{n-1},\kappa_n,\nu_{n-1})_{n=1}^{\infty}$ of the (C,F)-parameters of T and a coset $g\Gamma\in G/\Gamma$ such that*

$$\sum_{n=1}^{\infty}\kappa_n\big(\{c\in C_n\mid c\notin g\Gamma g^{-1}\}\big)<\infty.$$

An explicit formula for the factor mapping is obtained.

In §3 we consider *topological G-odometers* as the projective limits of homogeneous G-spaces G/Γ_n for a nested sequence $\Gamma_1\supsetneq\Gamma_2\supsetneq\cdots$ of cofinite subgroups in G such that $\bigcap_{n=1}^{\infty}\bigcap_{g\in G}g\Gamma_n g^{-1}=\{1_G\}$. By a *nonsingular G-odometer* we mean a topological G-odometer endowed with a G-quasiinvariant measure.

In §4 we study the odometer factors of nonsingular (C,F)-actions. The main result of the paper is the following (see Theorem 4.4).

Theorem. *Let (X,μ,T) be the nonsingular (C,F)-action of G associated with a sequence of (C,F)-parameters $\mathcal{T}$. Let O be the topological G-odometer* *defined on the projective space* $Y = \operatorname{proj}\lim_{n\to\infty} G/\Gamma_n$ *corresponding to a nested sequence* $\Gamma_1 \supsetneqq \Gamma_2 \supsetneqq \cdots$ *of cofinite subgroups in* G *such that* $\bigcap_{n=1}^{\infty} \Gamma_n = \{1_G\}$. *A measurable* G*-equivariant mapping* $\pi : X \to Y$ *exists if and only if there exist a telescoping* $\mathcal{T}' = (C_n, F_{n-1}, \kappa_n, \nu_{n-1})_{n=1}^{\infty}$ *of* $\mathcal{T}$ *and an element* $(g_n\Gamma_n)_{n=1}^{\infty} \in Y$ *such that*

$$\sum_{n=1}^{\infty} \kappa_n\big(\{c \in C_n \mid c \notin g_n\Gamma_n g_n^{-1}\}\big) < \infty.$$

An explicit formula for π *is obtained. Necessary and sufficient conditions for* π *to be an isomorphism of* (X, μ, T) *onto* $(Y, O, \mu \circ \pi^{-1})$ *are given in terms of* $\mathcal{T}'$.

As a corollary, criteria for a (C, F)-action to have an odometer factor or to be isomorphic to an odometer factor in terms of the underlying (C, F)-parameters are obtained in Corollaries 4.6 and 4.7.

In §5 we construct a rank-one action of the Heisenberg group $H_3(\mathbb{Z})$ which has an odometer factor but which is not isomorphic to any odometer action.

Розділ 1
Rank-one nonsingular actions of countable groups and (C, F)-construction

1.1. Nonsingular actions of rank one

Let G be a discrete infinite countable group. Let $T = (T_g)_{g \in G}$ be a free nonsingular action of G on a standard σ-finite non-atomic measure space $(X, \mathfrak{B}, \mu)$. By a *Rokhlin tower for* T we mean a pair (B, F), where $B \in \mathfrak{B}$ with $0 < \mu(B) < \infty$ and F is a finite subset of G with $1_G \in F$ such that

— the subsets $T_f B$, $f \in F$, are mutually disjoint,

— the Radon-Nikodym derivative $\dfrac{d\mu \circ T_f}{d\mu}$ is constant on B for each $f \in F$.

Given a Rokhlin tower (B, F), we let $X_{B,F} := \bigsqcup_{f \in F} T_f B \in \mathfrak{B}$. Of course, $\mu(X_{B,F}) < \infty$. By $\xi_{B,F}$ we mean the finite partition of $X_{B,F}$ into the subsets $T_f B$, $f \in F$. If $x \in T_f B$ then we set $O_{B,F}(x) := \{T_g x \mid g \in F f^{-1}\}$.

Definition 1.1. Let $\{1_G\} = F_0 \subset F_1 \subset F_2 \subset \cdots$ be an increasing sequence of finite subsets in G. We say that T is of *rank-one along* $(F_n)_{n=0}^{\infty}$ if there is a decreasing sequence $B_0 \supset B_1 \supset \cdots$ of subsets of positive measure in X such that (B_n, F_n) is a Rokhlin tower for T for each $n \in \mathbb{N}$ and

(i) $\xi_{B_n,F_n} \prec \xi_{B_{n+1},F_{n+1}}$ for each $n \geq 0$ and $\bigvee_{n=0}^{\infty} \xi_{B_n,F_n}$ is the partition of X into singletons (mod 0),

(ii) $\{T_g x \mid g \in G\} = \bigcup_{n=1}^{\infty} O_{B_n,F_n}(x)$ for a.e. $x \in X$.

It follows from (i) that $X_{B_0,F_0} \subset X_{B_1,F_1} \subset X_{B_2,F_2} \subset \cdots$ and $\bigcup_{n=0}^{\infty} X_{B_n,F_n} = X$. The piecewise constant property of the Radon-Nikodym derivative on the Rokhlin towers yields that:

(iii) if $T_c B_{n+1} \subset B_n$ for some $c \in F_{n+1}$ then

$$\frac{\mu(T_{fc}B_{n+1})}{\mu(T_f B_n)} = \frac{\mu(T_c B_{n+1})}{\mu(B_n)} \quad \text{for each } f \in F_n \text{ and } n \geq 0.$$

Proposition 1.2. *Let T satisfy (i) from Definition 1.1. Then T is ergodic. In particular, every rank-one nonsingular action is ergodic.*

Proof. Let two subsets $A_1, A_2 \in \mathfrak{B}$ be of positive measure. It follows from (i) that there are $n > 0$ and $f_1, f_2 \in F_n$ such that

$$\mu(A_1 \cap T_{f_1}B_n) > 0.9\mu(T_{f_1}B_n) \quad \text{and} \quad \mu(A_2 \cap T_{f_2}B_n) > 0.9\mu(T_{f_2}B_n).$$

As (B_n, F_n) is a Rokhlin tower, $T_{f_2 f_1^{-1}} T_{f_1} B_n = T_{f_2} B_n$ and

$$\frac{d\mu \circ T_{f_2 f_1^{-1}}}{d\mu}(x) = \frac{\mu(T_{f_2}B_n)}{\mu(T_{f_1}B_n)} \quad \text{at a.e. } x \in A_1.$$

It follows that

$$\begin{aligned}\mu\big(T_{f_2 f_1^{-1}} A_1 \cap T_{f_2} B_n\big) &= \mu\big(T_{f_2 f_1^{-1}}(A_1 \cap T_{f_1} B_n)\big) \\ &= \mu(A_1 \cap T_{f_1}B_n)\frac{\mu(T_{f_2}B_n)}{\mu(T_{f_1}B_n)} \\ &> 0.9\mu(T_{f_2 B_n}).\end{aligned}$$

Therefore $\mu(T_{f_2 f_1^{-1}} A_1 \cap T_{f_2} B_n \cap A_2) > 0.8\mu(T_{f_2}B_n)$. Hence, $\mu(T_{f_2 f_1^{-1}} A_1 \cap A_2) > 0$, as desired. $\square$

1.2. (C, F)-equivalence relations and nonsingular (C, F)-measures

Fix two sequences $(F_n)_{n\geq 0}$ and $(C_n)_{n\geq 1}$ of finite subsets in G such that $F_0 = \{1_G\}$ and for each $n > 0$,

$$\begin{aligned} &1_G \in F_n \cap C_n, \ \#C_n > 1, \\ &F_n C_{n+1} \subset F_{n+1}, \\ &F_n c \cap F_n c' = \varnothing \text{ if } c, c' \in C_{n+1} \text{ and } c \neq c'. \end{aligned} \tag{1-1}$$

We let $X_n := F_n \times C_{n+1} \times C_{n+2} \times \ldots$ and endow this set with the infinite product topology. Then X_n is a compact Cantor space. The mapping

$$X_n \ni (f_n, c_{n+1}, c_{n+2} \ldots) \mapsto (f_n c_{n+1}, c_{n+2}, \ldots) \in X_{n+1}$$

is a continuous embedding of X_n into X_{n+1}. Therefore the topological inductive limit X of the sequence $(X_n)_{n\geq 0}$ is well defined. Moreover, X is a locally compact Cantor space. Given a subset $A \subset F_n$, we let

$$[A]_n := \{x = (f_n, c_{n+1}, \ldots) \in X_n, f_n \in A\}$$

and call this set an *n-cylinder* in X. It is open and compact in X. For brevity, we will write $[f]_n$ for $[\{f\}]_n$ for an element $f \in F_n$.

We remind that two points $x = (f_n, c_{n+1}, \ldots)$ and $x' = (f'_n, c'_{n+1}, \ldots)$ of X_n are *tail equivalent* if there is $N > n$ such that $c_l = c'_l$ for each $l > N$. We thus obtain the tail equivalence relation on X_n.

Definition 1.3 ([5]). The (C, F)*-equivalence relation (or the tail equivalence relation)* $\mathcal{R}$ on X is defined as follows: for each $n \geq 0$, the restriction of $\mathcal{R}$ to X_n is the tail equivalence relation on X_n.

The following properties of $\mathcal{R}$ are easy to check:

— Each $\mathcal{R}$-class is countable.

— $\mathcal{R}$ is *minimal*, i.e. the $\mathcal{R}$-class of every point is dense in X.

— $\mathcal{R}$ is *hyperfinite*, i.e. there is a sequence $(\mathcal{S}_n)_{n=1}^{\infty}$ of subrelations of $\mathcal{R}$ such that $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \cdots$, $\bigcup_{n=1}^{\infty} \mathcal{S}_n = \mathcal{R}$ and $\#\mathcal{S}_n(x) < \infty$ for each $x \in X$ and $n \geq 0$. Indeed, we can define $\mathcal{S}_n$ by the following: $(x, y) \in \mathcal{S}_n$ if either $x, y \notin X_n$ and $x = y$ or $x = (f_n, c_{n+1}, \dots) \in X_n, y = (f'_n, c'_{n+1}, \dots) \in X_n$ and $c_m = c'_m$ for all $m > n$.

We remind that the *full group* $[\mathcal{R}$ *of* $\mathcal{R}$ is the group of all Borel bijections $\gamma : X \to X$ such that $(x, \gamma x) \in \mathcal{R}$ for each $x \in X$. A Borel measure μ on X is called $\mathcal{R}$-quasi-invariant if $\mu \circ \gamma \sim \mu$ for each $\gamma \in [\mathcal{R}]$. Then there is a Borel mapping $\rho_\mu : \mathcal{R} \to \mathbb{R}_+^*$ such that

$$\rho_\mu(x, y)\rho_\mu(y, z) = \rho_\mu(x, z) \qquad \text{for all } (x, y), (y, z) \in \mathcal{R}$$

and $\rho_\mu(\gamma x, x) = \dfrac{d\mu \circ \gamma}{d\mu}(x)$ at a.e. $x \in X$ for each $\gamma \in [\mathcal{R}]$. The mapping ρ_μ is called *the Radon-Nikodym cocycle of* $(\mathcal{R}, \mu)$.

Suppose that for each $n \in \mathbb{N}$, a non-degenerated probability measure κ_n on C_n is given. We now let $\mu_0 := \nu_0 \otimes \kappa_1 \otimes \kappa_2 \otimes \cdots$, where ν_0 is the Dirac measure supported at 1_G. Then μ_0 is an $(\mathcal{R} \restriction X_0)$-quasi-invariant probability on X_0. Of course, μ_0 is non-atomic if and only if

$$\prod_{n>0} \max_{c \in C_n} \kappa_n(c) = 0. \tag{1-2}$$

By the Kolmogorov 0-1 law, $(\mathcal{R} \restriction X_0)$ is ergodic on the probability space (X_0, μ_0). There are many ways to extend μ_0 to an $\mathcal{R}$-quasiinvariant measure on X. However all such measures will be mutually equivalent. Select for each

$n \in \mathbb{N}$, a non-degenerated finite measure ν_n on F_n such that

$$\nu_{n+1}(fc) = \nu_n(f)\kappa_{n+1}(c) \quad \text{for each } f \in F_n \text{ and } c \in C_{n+1}. \tag{1-3}$$

It is often convenient to consider ν_n and κ_n as finite measures on G supported on F_n and C_n respectively. Then (1-3) means that $\nu_{n+1} \restriction F_n C_{n+1} = \nu_n * \kappa_{n+1}$, where the symbol $*$ means the convolution. We now define a Borel measure μ on X by setting

$$\mu([f]_n) := \nu_n(f), \quad \text{for each } g \in F_n \text{ and every } n \in \mathbb{N}.$$

It is straightforward to verify that μ is a well defined σ-finite Radon measure. Moreover, μ is $\mathcal{R}$-quasi-invariant and

$$\rho_\mu(x,y) = \frac{\nu_n(f_n)}{\nu_n(f_n')} \prod_{m>n} \frac{\kappa_m(c_m)}{\kappa_m(c_m')},$$

whenever $x = (f_n, c_{n+1}, \dots)$ and $y = (f_n', c_{n+1}', \dots)$ are $\mathcal{R}$-equivalent points that belong to $X_n = F_n \times C_{n+1} \times C_{n+2} \times \cdots$ for some $n > 0$.

The following definition extends [3], Definition 4.2, where the case of Abelian G was considered.

Definition 1.4. If (1-2) and (1-3) hold then we call μ *the (C,F)-measure on X determined by $(\kappa_n)_{n=1}^\infty$ and $(\nu_n)_{n=0}^\infty$.*

Consider another sequence $(\nu_n')_{n=0}^\infty$ of non-degenerated measures on $(F_n)_{n=0}^\infty$ respectively (in n) such that ν_0' is the Dirac measure supported at 1_G and $\nu_{n+1}'(fc) = \nu_n'(f)\kappa_{n+1}(c)$ for each $f \in F_n$ and $c \in C_{n+1}$ for each $n > 0$. Then the (C,F)-measure μ' determined by $(\kappa_n)_{n=1}^\infty$ and $(\nu_n')_{n=0}^\infty$ is equivalent to μ and

$$\frac{d\mu'}{d\mu}(x) = \frac{\nu_n'(f_n)}{\nu_n(f_n)} \quad \text{if } x = (f_n, \dots) \in X_n.$$

Another useful observation is that given $(\kappa_n)_{n=1}^\infty$, we can always find $(\nu_n)_{n=0}^\infty$

satisfying (1-3). Thus, the equivalence class of a nonsingular (C, F)-measure is completely determined by $(\kappa_n)_{n=1}^{\infty}$ alone. In particular, we may always replace a σ-finite nonsingular (C, F)-measure with an equivalent finite nonsingular (C, F)-measure.

Remark 1.5. We note that $\mathcal{R}$ is *Radon uniquely ergodic*, i.e. there is a unique $\mathcal{R}$-invariant Radon measure ξ on X such that $\xi(X_0) = 1$. We call it the *Haar measure for* $\mathcal{R}$. It is σ-finite. Let k_n be the equidistribution on C_n and let $\nu_n(f) = \prod_{k=1}^{n} \kappa_k(1_G)$ for each $f \in F_n$ and $n \geq 0$. Then (1-2) and (1-3) holds for $(\kappa_n)_{n=1}^{\infty}$ and $(\nu_n)_{n=0}^{\infty}$. Of course, the Haar measure for $\mathcal{R}$ is a (C, F)-measure determined by $(\kappa_n)_{n=1}^{\infty}$ and $(\nu_n)_{n=0}^{\infty}$. The Haar measure is finite if and only if

$$\prod_{n=1}^{\infty} \frac{\#F_{n+1}}{\#F_n \#C_{n+1}} < \infty.$$

It is easy to verify that $\mathcal{R}$ is conservative and ergodic on the σ-finite measure space (X, μ). This means that for each $\mathcal{R}$-invariant subset $A \subset X$, either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

1.3. Nonsingular (C, F)-actions

Nonsingular (C, F)-actions were defined in [3] and [4] for Abelian groups only. We extend this definition to arbitrary countable groups. Given $g \in G$, let

$$X_n^g := \{(f_n, c_{n+1}, c_{n+2}, \dots) \in X_n \mid g f_n \in F_n\}.$$

Then X_n^g is a compact open subset of X_n and $X_n^g \subset X_{n+1}^g$. Hence the union $X^g := \bigcup_{n \geq 0} X_n^g$ is an open subset of X. Let $X^G := \bigcap_{g \in G} X^g$. Then X^G is a G_δ-subset of X. Hence X^G is Polish and totally disconnected in the induced topology. Given $g \in G$ and $x \in X_G$, there is $n > 0$ such that $x =$

$(f_n, c_{n+1}, \dots) \in X_n$ and $gf_n \in F_n$. We now let $T_g x := (gf_n, c_{n+1}, \dots) \in X_n \subset X$. It is straightforward to verify that

(i) $T_g x \in X^G$,

(ii) the mapping $T_g : X^G \ni x \mapsto T_g x \in X^G$ is a homeomorphism of X^G and

(iii) $T_g T_{g'} = T_{gg'}$ for all $g, g' \in G$.

Hence $T := (T_g)_{g \in G}$ is a continuous G-action on X^G.

Definition 1.6 ([5]). The action T is called *the topological* (C, F)*-action of* G *associated with* $(C_n, F_{n-1})_{n=1}^{\infty}$.

This action is free. The subset X^G is $\mathcal{R}$-invariant. The T-orbit equivalence relation coincides with the restriction of $\mathcal{R}$ to X^G.

Proposition 1.7 ([5], Proposition 1.2). $X^G = X$ *if and only if for each* $g \in G$ *and* $n > 0$, *there is* $m > n$ *such that*

$$gF_n C_{n+1} C_{n+2} \cdots C_m \subset F_m. \tag{1-4}$$

Thus, if (1-4) holds then T is a minimal continuous G-action on a locally compact Cantor space X. Moreover, T is *Radon uniquely ergodic*, i.e. there exists a unique T-invariant Radon measure ξ on X such that $\xi(X_0) = 1$.

From now on, T is a topological (C, F)-action of G on X^G and μ is the nonsingular (C, F)-measure on X determined by $(\kappa_n)_{n=1}^{\infty}$ and $(\nu_n)_{n=0}^{\infty}$ satisfying (1-2) and (1-3). Since X^G is $\mathcal{R}$-invariant, we obtain that either $\mu(X^G) = 0$ or $\mu(X \setminus X^G) = 0$. In the latter case T is μ-nonsingular, conservative and ergodic.

Proposition 1.8. *The following are equivalent.*

(i) $\mu(X \setminus X^G) = 0$.

(ii) For each $g \in G$ and every $n \geq 0$,

$$\lim_{m\to\infty} \nu_m\big((F_nC_{n+1}C_{n+2}\cdots C_m)\cap g^{-1}F_m\big) = \nu_n(F_n).$$

(iii) For each $g \in G$,

$$\lim_{m\to\infty} \kappa_1 * \cdots * \kappa_m(g^{-1}F_m) = 1.$$

Proof. (i)$\Leftrightarrow$(ii) Since $\mu(X \setminus X^G) = 0$ if and only if $\mu(X_n \cap X_m^g) \to \mu(X_n)$ as $m \to \infty$ for each $g \in G$ and $n \geq 0$, it suffices to note that

$$\begin{aligned}
\mu(X_n \cap X_m^g) &= \mu([F_n]_n \cap [F_m \cap g^{-1}F_m]_m) \\
&= \mu([F_nC_{n+1}\cdots C_m]_m \cap [F_m \cap g^{-1}F_m]_m) \\
&= \mu([F_nC_{n+1}\cdots C_m \cap F_m \cap g^{-1}F_m]_m) \\
&= \nu_m((F_nC_{n+1}\cdots C_m)\cap g^{-1}F_m)
\end{aligned}$$

and $\mu(X_n) = \mu([F_n]_n) = \nu_n(F_n)$.

(ii)$\Rightarrow$(iii) We set $\kappa_{1,m} := \kappa_1 * \cdots * \kappa_m$. Then

$$\begin{aligned}
\kappa_{1,m}\big((C_1\cdots C_m)\setminus g^{-1}F_m\big) &= \nu_m\big((F_0C_1\cdots C_m)\setminus g^{-1}F_m\big) \\
&= \nu_m(F_0C_1\cdots C_m) - \nu_m\big((F_0C_1\cdots C_m)\cap g^{-1}F_m\big) \\
&= \nu_0(F_0) - \nu_m\big((F_0C_1\cdots C_m)\cap g^{-1}F_m\big).
\end{aligned}$$

Hence, $\lim_{m\to\infty} \kappa_{1,m}\big((C_1\cdots C_m)\setminus g^{-1}F_m\big) = 0$ according to (ii). As $\kappa_{1,m}$ is supported on $C_1\cdots C_m$, it follows that

$$\begin{aligned}
\lim_{m\to\infty} \kappa_{1,m}(g^{-1}F_m) &= \lim_{m\to\infty} \kappa_{1,m}\big((C_1\cdots C_m)\cap g^{-1}F_m\big) \\
&= \lim_{m\to\infty} \kappa_{1,m}(C_1\cdots C_m) = 1,
\end{aligned}$$

as desired.

(iii)$\Rightarrow$(i) Fix $g \in G$. Take arbitrary $n \geq 0$ and $f \in F_n$. Then it follows from (iii) that for μ-a.e. $x = (1_G, c_{n+1}, c_{n+2},) \in [1_G]_n$, there exists $m > 0$ such that $gfc_{n+1}\cdots c_m\in F_m$. This means that for μ-a.e. $y=(f_n,c_{n+1},c_{n+2},\dots)\in X_n$,

$$gf_nc_{n+1}\cdots c_m\in F_m\quad\text{eventually in } m,$$

i.e. $y\in X^g$. Hence $\mu(X\setminus X^g)=0$ and (i) follows. □

In the case where μ is the Haar measure for $\mathcal{R}$, the equivalence (i)$\Leftrightarrow$(ii) of Proposition 1.8 was proved in [5].

Definition 1.9. If $\mu(X\setminus X^G)=0$ then the dynamical system (X,μ,T) (or simply T) is called *the nonsingular* (C,F)*-action associated with* $(C_n,F_{n-1},\kappa_n,\nu_{n-1})_{n=1}^\infty$.

From now on we consider only the case where $\mu(X\setminus X^G)=0$. As $X=X^G$ mod 0, we will assume that T is defined on the entire space X. Then for each n and every two elements $g,h\in F_n$, we have that $T_{hg^{-1}}[g]_n=[h]_n$ and the Radon-Nikodym derivative of the transformation $T_{hg^{-1}}$ is constant on the subset $[g]_n$. More precisely, this constant equals $\dfrac{\nu_n(h)}{\nu_n(g)}$.

We now state the theorem from the [8].

Theorem 1.10. *Each nonsingular* (C,F)*-action is of rank one. Conversely, each rank-one nonsingular* G*-action is isomorphic (via a measure preserving isomorphism) to a* (C,F)*-action.*

1.4. Telescopings

Let a sequence $\mathcal{T}=(C_n,F_{n-1},\kappa_n,\nu_{n-1})_{n=1}^\infty$ satisfy (1-1)–(1-3) and Proposition 1.8(ii). Denote by $T=(T_g)_{g\in G}$ the (C,F)-action of G on X associated with $\mathcal{T}$. Let μ stand for the the nonsingular (C,F)-measure on X determined by $(\kappa_n)_{n=1}^\infty$ and $(\nu_n)_{n=0}^\infty$.

Given a strictly increasing infinite sequence of integers $\boldsymbol{l}=(l_n)_{n=0}^\infty$ such that

$l_0 = 0$, we let

$$\widetilde{F}_n := F_{l_n}, \quad \widetilde{C}_{n+1} := C_{l_n+1} \cdots C_{l_{n+1}}, \quad \widetilde{\nu}_n := \nu_n, \quad \widetilde{\kappa}_{n+1} := \kappa_{l_n+1} * \cdots * \kappa_{l_{n+1}}.$$

for each $n \geq 0$.

Definition 1.11. s We call the sequence $\widetilde{\mathcal{T}} := (\widetilde{C}_n, \widetilde{F}_{n-1}, \widetilde{\kappa}_n, \widetilde{\nu}_{n-1})_{n=1}^{\infty}$ the $\boldsymbol{l}$-*telescoping* of $\mathcal{T}$.

It is easy to check that $\widetilde{\mathcal{T}}$ satisfies (1-1)–(1-3) and Proposition 1.8(ii). Hence a nonsingular (C, F)-action $\widetilde{T} = (\widetilde{T}_g)_{g \in G}$ of G associated with $\widetilde{\mathcal{T}}$ is well defined. Let $\widetilde{X}$ denote the space of $\widetilde{T}$ and let $\widetilde{\mu}$ denote the nonsingular (C, F)-measure on $\widetilde{X}$ determined by $(\widetilde{\kappa}_n)_{n=1}^{\infty}$ and $(\widetilde{\nu}_n)_{n=0}^{\infty}$. There is a *canonical* measure preserving isomorphism $\iota_{\boldsymbol{l}}$ of (X, μ) onto $(\widetilde{X}, \widetilde{\mu})$ that intertwines T with $\widetilde{T}$. Indeed, if $x \in X$ then we select the smallest $n \geq 0$ such that $x = (f_{l_n}, c_{l_n+1}, c_{l_n+1}, \dots) \in X_{l_n}$. Let

$$\iota_{\boldsymbol{l}}(x) := (f_{l_n}, c_{l_n+1} \cdots c_{l_{n+1}}, c_{l_{n+1}+1} \cdots c_{l_{n+2}}, \dots) \in \widetilde{X}_n \subset \widetilde{X},$$

where $\widetilde{X}_n = \widetilde{F}_n \times \widetilde{C}_{n+1} \times \widetilde{C}_{n+2} \times \cdots$. It is a routine to verify that $\iota_{\boldsymbol{l}}$ is a homeomorphism of X onto $\widetilde{X}$ such that $\iota_{\boldsymbol{l}} T_g = \widetilde{T}_g \iota_{\boldsymbol{l}}$ for each $g \in G$, as desired.

Let $\boldsymbol{l} = (l_n)_{n=0}^{\infty}$ and $\boldsymbol{m} = (m_n)_{n=0}^{\infty}$ be two strictly increasing sequences of integers such that $l_0 = m_0 = 0$. If $\widetilde{\mathcal{T}}$ is the $\boldsymbol{l}$-telescoping of $\mathcal{T}$ and $\mathcal{S}$ is the $\boldsymbol{m}$-telescoping of $\widetilde{\mathcal{T}}$ then $\mathcal{S}$ is the $\boldsymbol{l} \circ \boldsymbol{m}$-telescoping of $\mathcal{T}$, where $\boldsymbol{l} \circ \boldsymbol{m} := (l_{m_n})_{n=1}^{\infty}$ and $\iota_{\boldsymbol{m}} \circ \iota_{\boldsymbol{l}} = \iota_{\boldsymbol{l} \circ \boldsymbol{m}}$.

1.5. Nonsingular $\mathbb{Z}$-actions of rank one along intervals and (C, F)-construction

Let $G = \mathbb{Z}$. Suppose that a sequence $\mathcal{T} = (C_n, F_{n-1}, \kappa_n, \nu_{n-1})_{n=1}^{\infty}$ satisfies (1-1)–(1-3) and Proposition 1.8(ii) and there is a sequence $(h_n)_{n=0}^{\infty}$ of positive integers such that $F_n = \{0, 1, \dots, h_n - 1\}$. Denote by (X, μ, T) the (C, F)-dynamical system associated with $\mathcal{T}$.

We now show how to obtain (X, μ, T) via the classical inductive geometric cutting-and-stacking in the case of $\mathbb{Z}$-actions. On the initial step of the construction we define a column Y_0 consisting of a single interval $[0, 1)$ equipped with Lebesgue measure. Assume that at the n-th step we have a column

$$Y_n = \{I(i, n) \mid i = 0, \dots, h_n - 1\}$$

consisting of of disjoint intervals $I(i, n) \subset \mathbb{R}$ such that $\bigsqcup_{i=0}^{h_n-1} I(i, n) = [0, \nu_n(F_n))$. Then we define a continuous *n-th column mapping*

$$T^{(n)} : \big[0, \nu_n(F_n)\big) \setminus I(h_n - 1, n) \to \big[0, \nu_n(F_n)\big) \setminus I(0, n)$$

such that $T^{(n)} \upharpoonright I(i, n)$ is the orientation preserving affine mapping of $I(i, n)$ onto $I(i+1, n)$ for $i = 0, \dots, h_n-2$. It is convenient to think of $I(i, n)$ as a *level* of Y_n. The levels may be of different length but they are parallel to each other and the i-th level is above the j-th level if $i > j$. The n-th column mapping moves every level, except the highest one, one level up. By the move here we mean the orientation preserving affine mapping. On the highest level, $T^{(n)}$ is not defined.

On the $(n + 1)$-th step of the construction, we first cut each $I(i, n)$ into subintervals $I(i + c, n + 1)$, $c \in C_{n+1}$, such that $I(i + c, n + 1)$ is from the left of $I(i + c', n + 1)$ whenever $c < c'$ and

$$\frac{\text{the length of } I(i + c, n + 1)}{\text{the length of } I(i, n)} = \kappa_{n+1}(c) \quad \text{for each } c \in C_{n+1}.$$

Hence, we obtain that $\bigsqcup_{j \in F_n + C_{n+1}} I(j, n + 1) = \big[0, \nu_n(F_n)\big)$. Next, we cut the interval $\big[\nu_n(F_n), \nu_{n+1}(F_{n+1})\big)$ into subintervals $I(j, n + 1)$, $j \in F_{n+1} \setminus (F_n +$

C_{n+1}), such that the length of $I(j,n+1)$ is $\nu_{n+1}(j)$ for each j. These new levels are called *spacers*. Thus, we obtain a new column $Y_{n+1} = \{I(j,n+1) \mid j \in F_{n+1}\}$ with $\bigsqcup_{i\in F_{n+1}} I(i,n+1) = [0,\nu_{n+1}(F_{n+1}))$. If an element $c \in C_{n+1}$ is not maximal in C_{n+1} then we denote by c^+ the least element of C_{n+1} that is greater than c. We define a *spacer mapping* $s_{n+1} : C_{n+1} \to \mathbb{Z}_+$ by setting

$$s_{n+1}(c) := \begin{cases} c^+ - c - h_n & \text{if } c \neq \max C_{n+1}, \\ h_{n+1} - c - h_n & \text{if } c = \max C_{n+1}. \end{cases}$$

The subcolumn $Y_{n,c} := \{I(i+c,n+1) \mid i \in F_n\} \subset Y_{n+1}$ is called *the c-copy of* Y_n, $c \in C_{n+1}$. Thus, Y_{n+1} consists of $\#C_{n+1}$ copies of Y_n and spacers between them and above the highest copy of Y_n. More precisely, there are exactly $s_{n+1}(c)$ spacers above the c-copy of Y_n in Y_{n+1}. The $(n+1)$-th column mapping

$$T^{(n+1)} : \big[0,\nu_{n+1}(F_{n+1})\big) \setminus I(h_{n+1}-1,n+1) \to \big[0,\nu_{n+1}(F_{n+1})\big) \setminus I(0,n+1)$$

is defined in a similar way as $T^{(n)}$. Of course,

$$T^{(n+1)} \restriction \big(\big[0,\nu_n(F_n)\big) \setminus I(h_n-1,n)\big) = T^{(n)}.$$

Passing to the limit as $n \to \infty$, we obtain a well-defined nonsingular (piecewise affine) transformation Q of the interval $[0, \lim_{n\to\infty} \nu_n(F_n)) \subset \mathbb{R}$ equipped with Lebesgue measure such that

$$Q \restriction \Big(\big[0,\nu_n(F_n)\big) \setminus I(h_n-1,n)\Big) = T^{(n)} \qquad \text{for each } n \in \mathbb{N}.$$

It is possible that $\lim_{n\to\infty} \nu_n(F_n) = \infty$ and then T is defined on $[0,+\infty)$. Of course, this transformation (or, more precisely, the $\mathbb{Z}$-action generated by Q) is isomorphic to (X,μ,T). The according nonsingular isomorphism is generated by the following correspondence:

$$X \supset [i]_n \longleftrightarrow I(i,n) \subset \big[0, \lim_{n\to\infty} \nu_n(F_n)\big), \quad i \in F_n, n \in \mathbb{N}.$$

Without loss of generality we may assume $s_n(\max C_n) = 0$ for each $n > 0$ (see, for instance, [6]). This means that there are no spacers over the highest copy of Y_n in Y_{n+1}.

Розділ 2
Finite factors of nonsingular (C, F)-actions

Let a sequence $\mathcal{T} = (C_n, F_{n-1}, \kappa_n, \nu_{n-1})_{n=1}^{\infty}$ satisfy (1-1)–(1-3) and Proposition 1.8(ii) and let $\lim_{n\to\infty} \nu_n(F_n) < \infty$. Let Γ be a cofinite subgroup of G. We consider the left coset space G/Γ as a homogeneous G-space on which G acts by left translations. It is obvious that for each coset $g\Gamma \in G/\Gamma$, the subgroup $g\Gamma g^{-1} \subset G$ is the stabilizer of $g\Gamma$ in G.

Definition 2.1. Given a coset $g\Gamma \in G/\Gamma$, we say that $\mathcal{T}$ is *compatible with* $g\Gamma$ if

$$\sum_{n=1}^{\infty} \kappa_n\big(\{c \in C_n \mid c \notin g\Gamma g^{-1}\}\big) < \infty.$$

Denote by (X, μ, T) the (C, F)-action of G associated with $\mathcal{T}$. Then $\mu(X) = \lim_{n\to\infty} \nu_n(F_n) < \infty$. For a point $x = (c_1, c_2, \dots) \in X_0 = C_1 \times C_2 \times \cdots$, we let

$$\pi_{(\mathcal{T}, g\Gamma)}(x) := \lim_{n\to\infty} c_1 c_2 \cdots c_n g\Gamma \in G/\Gamma$$

whenever this limit exists[1]. It follows from the Borel-Cantelli lemma that if $\mathcal{T}$ is compatible with $g\Gamma \in G/\Gamma$ then $\pi_{(\mathcal{T}, g\Gamma)}(x)$ is well defined for μ-a.e. $x \in X_0$. It is straightforward to verify that for each $h \in G$,

$$\pi_{(\mathcal{T}, g\Gamma)}(T_h x) = h\pi_{(\mathcal{T}, g\Gamma)}(x)$$

[1] The quotient space G/Γ is endowed with the discrete topology.

whenever $\pi_{(\mathcal{T},g\Gamma)}(x)$ is well defined and $T_h x \in X_0$. It follows that the mapping

$$\pi_{(\mathcal{T},g\Gamma)} : X_0 \ni x \mapsto \pi_{(\mathcal{T},g\Gamma)}(x) \in G/\Gamma$$

extends uniquely (mod 0) to a measurable G-equivariant mapping from X to G/Γ. We denote the extension by the same symbol $\pi_{(\mathcal{T},g\Gamma)}$. It is routine to verify that if $x = (f_n, c_{n+1}, c_{n+2}, \dots) \in X_n$ for some $n \in \mathbb{N}$ then

$$\pi_{(\mathcal{T},g\Gamma)}(x) := \lim_{m\to\infty} f_n c_{n+1} c_{n+2} \cdots c_m g\Gamma. \tag{2-1}$$

Definition 2.2. We call $\pi_{(\mathcal{T},g\Gamma)}$ *the* $(\mathcal{T}, g\Gamma)$*-factor mapping* for T.

We need some notation. Given $1 < n < m$, we denote the subset $C_n \cdots C_m$ of G by $C_{n,m}$. Let $\kappa_{n,m}$ stand for the probability measure $\kappa_n * \cdots * \kappa_m$. It is supported on $C_n \cdots C_m$. We now state the main result of the section.

Theorem 2.3. *The following are equivalent.*

(i) There is a measurable factor map $\tau : X \to G/\Gamma$, i.e. for each $g \in G$,

$$\tau(T_g x) = g\tau(x) \quad \textit{at } \mu\text{-a.e. } x \in X$$

(ii) There exists a sequence $(g_n)_{n>0}$ of elements of G such that

$$\lim_{N\to\infty} \sup_{m>n\geq N} \kappa_{n+1,m}\big(\{c \in C_{n+1,m} \mid c \notin g_n^{-1}\Gamma g_m\}\big) = 0.$$

(iii) There exist a coset $g_0\Gamma \in G/\Gamma$ and a $g_0\Gamma$-compatible telescoping of $\mathcal{T}$.

It follows that T has no factors isomorphic to G/Γ if and only if there is no telescoping of $\mathcal{T}$ compatible with $g\Gamma g^{-1}$ for any $g \in G$.

Proof. (i) $\Longrightarrow$ (ii) Let $Y_j := \tau^{-1}(j)$, for each $j \in G/\Gamma$. Then $X = \displaystyle\bigsqcup_{j\in G/\Gamma} Y_j$. Consider n large so that $\mu(Y_j \cap X_n) > 0$ for each $j \in G/\Gamma$. Let $g, h \in F_n$. Since

τ is G-equivariant, it follows that $T_{hg^{-1}}Y_j = Y_{hg^{-1}j}$ and hence

$$T_{hg^{-1}}([g]_n \cap Y_j) = [h]_n \cap Y_{hg^{-1}j}. \tag{2-2}$$

For each $j \in G/\Gamma$ and $g \in F_n$, let

$$d_{n,g}(j) := \mu([g]_n \cap Y_j)/\mu([g]_n).$$

Then, the set $\{d_{n,g}(j) \mid j \in G/\Gamma\} \subset (0,1)$ does not depend on $g \in F_n$. Indeed, for each $h \in F_n$, the Radon-Nikodym derivative of the transformation $T_{gh^{-1}}$ is constant on $[h]_n$ and we deduce from (2-2) that

$$d_{n,h}(j) := \frac{\mu([h]_n \cap Y_j)}{\mu([h]_n)} = \frac{\mu([g]_n \cap Y_{gh^{-1}j})}{\mu([g]_n)} = d_{n,g}(gh^{-1}j). \tag{2-3}$$

Hence $\{d_{n,h}(j) : j \in G/\Gamma\} = \{d_{n,g}(j) : j \in G/\Gamma\}$. Let

$$\delta_n := \max_{j \in G/\Gamma} d_{n,g}(j).$$

We claim that $\delta_m \to 1$ as $m \to \infty$. Indeed, for $m \geq n$, let $\mathcal{P}_m$ denote the finite σ-algebra generated by the m-cylinders (which are compact and open subsets of X) that are contained in X_n. Then $\mathcal{P}_n \subset \mathcal{P}_{n+1} \subset \cdots$ and $\bigvee_{m>n} \mathcal{P}_m$ is the entire Borel σ-algebra on X_n. Hence, for each $j \in G/\Gamma$, there is $g_m \in F_m$ such that

$$\frac{\mu([g_m]_m \cap Y_j)}{\mu([g_m]_m)} \to 1 \quad \text{as } m \to \infty.$$

This implies that $\delta_m \to 1$ as $m \to \infty$, as claimed. In what follows we consider n large so that $\delta_n > 0.9$. Then for each $g \in F_n$, there is a unique Γ-coset $j_n(g) \in G/\Gamma$ such that $\delta_n = d_{n,g}(j_n(g))$. It is convenient to consider G/Γ as a set of *colors*. Then $j_n(g)$ is *the dominating color* on $[g]_n$. It follows from (2-3) that $j_n(g) = gh^{-1}j_n(h)$, for all $g, h \in F_n$. Choose $g_n \in F_n$ such that $j_n(g_n) = \Gamma$.

Then $j_n(g) = gg_n^{-1}\Gamma$ for each $g \in F_n$. Given $\epsilon < \dfrac{1}{2}$, there is $N > 0$ such that $\delta_n > 1 - \epsilon^2$ for all $n > N$. Hence for all $m > n > N$,

$$(1 - \epsilon^2)\mu([1_G]_n) < \mu([1_G]_n \cap Y_{j_n(1_G)}). \tag{2-4}$$

We remind that $[1_G]_n = \bigsqcup_{c \in C_{n+1,m}} [c]_m$. Let

$$D := \{c \in C_{n,m} \mid \mu([c]_m \cap Y_{j_n(1_G)}) > (1 - \epsilon)\mu([c]_m)\}.$$

It follows from (2-4) that

$$\begin{aligned}(1 - \epsilon^2)\mu([1_G]_n) &< \sum_{c \in D} \mu([c]_m) + (1 - \epsilon) \sum_{c \in C_{n+1,m} \setminus D} \mu([c]_m) \\ &= \sum_{c \in D} \mu([c]_m) + (1 - \epsilon)\left(\mu([1_G]_n) - \sum_{c \in D} \mu([c]_m)\right).\end{aligned}$$

This yields that $\sum_{c \in D} \mu([c]_m) > (1 - \epsilon)\mu([1_G]_n)$ or, equivalently,

$$\sum_{c \in D} \kappa_{n+1,m}(c) > (1 - \epsilon)\kappa_n(1_G). \tag{2-5}$$

By the definition of D, an element $c \in C_{n+1,m}$ belongs to D if and only if $j_n(1_G)$ is the dominating color on $[c]_m$, i.e. $j_m(c) = j_n(1_G)$. Therefore, $cg_m^{-1}\Gamma = g_n^{-1}\Gamma$, i.e. $c \in g_n^{-1}\Gamma g_m$. Hence (2-5) yields that

$$\kappa_{n+1,m}\big(\{c \in C_{n+1,m} \mid c \notin g_n^{-1}\Gamma g_m\}\big) \le \epsilon$$

and (ii) follows.

(ii) $\Longrightarrow$ (iii) As Γ is cofinite and (ii) holds, there exist an increasing sequence $0 = q_0 < q_1 < q_2 < \cdots$ of positive integers and an element $g_0 \in G$ such that

$g_n^{-1} \in g_0\Gamma$ and

$$\kappa_{q_n+1,q_{n+1}}\big(\{c \in C_{q_n+1,q_{n+1}} \mid c \notin g_0\Gamma g_0^{-1}\}\big) < 2^{-n}$$

for each $n \in \mathbb{N}$. Let $\boldsymbol{q} := (q_n)_{n=0}^{\infty}$. Then $g_0\Gamma$ is compatible with the $\boldsymbol{q}$-telescoping of $\mathcal{T}$. This implies (iii).

(iii) $\Longrightarrow$ (i) Denote by $\widetilde{\mathcal{T}}$ the $\boldsymbol{q}$-telescoping of $\mathcal{T}$. Then $\pi_{(\widetilde{\mathcal{T}},g_0\Gamma)} \circ \iota_{\boldsymbol{q}}$ is a factor mapping of T onto G/Γ.

Thus, the first statement of the theorem is proved completely. The second statement follows from the first one and a simple observation that given two subgroups Γ and Γ' of G, the corresponding G-actions by left translations on G/Γ and G/Γ' are isomorphic if and only if Γ and Γ' are conjugate. $\square$

The following important remark will be used essentially in the proof of the main result of §3.

Remark 2.4. In fact, we obtained more than what is stated in Theorem 2.3. We proved indeed that given a factor mapping $\tau : X \to G/\Gamma$, there exist a coset $g_0\Gamma$ and a $g_0\Gamma$-compatible $\boldsymbol{q}$-telescoping $\widetilde{\mathcal{T}}$ of $\mathcal{T}$ such that $\tau = \pi_{(\widetilde{\mathcal{T}},g_0\Gamma)} \circ \iota_{\boldsymbol{q}}$. To explain this fact, we use below the notation from the proof of Theorem 2.3. For $n \in \mathbb{N}$, let

$$X'_n := \bigsqcup_{f_n \in F_n} \big([f_n]_n \cap Y_{j_n(f_n)}\big) \subset X_n.$$

We remind that $\mu(X) < \infty$. Since $\delta_n \to 1$ and $\mu(X_n) \to \mu(X)$ as $n \to \infty$, it follows that $\mu(X'_{q_n}) \to \mu(X)$ as $n \to \infty$. We can assume (passing to a subsequence of $(q_n)_{n=1}^{\infty}$ if needed) that $\sum_{n=1}^{\infty} \mu(X \setminus X'_{q_n}) < \infty$. Then, the Borel-Cantelli lemma yields that for a.e. $x \in X$, we have that $x \in X'_{q_n}$ eventually in n. Hence, for a.e. $x \in X$,

$$\tau(x) = \lim_{m\to\infty} j_{q_m}(f_{q_m}) = \lim_{m\to\infty} f_{q_m} g_{q_m}^{-1}\Gamma = \lim_{m\to\infty} f_{q_n} g_0\Gamma, \tag{2-6}$$

where f_{q_m} is the first coordinate of x in X_{q_m}, i.e. $x = (f_{q_m}, c_{q_m+1}, \dots) \in X_{q_m}$. On the other hand, it follows from (2-1) and (2-2) that

$$\pi_{(\widetilde{\mathcal{T}}, g_0\Gamma)}(\iota_{\boldsymbol{q}}(x)) = \lim_{m\to\infty} f_{q_m} g_0 \Gamma \tag{2-7}$$

at a.e. $x \in X$. Therefore, (2-6) and (2-7) yield that $\tau = \pi_{(\widetilde{\mathcal{T}}, g_0\Gamma)} \circ \iota_{\boldsymbol{q}}$ almost everywhere, as desired.

Розділ 3
Nonsingular odometer actions of residually finite groups

From now on G is residually finite. Then there is a decreasing sequence $\Gamma_1 \supsetneq \Gamma_2 \supsetneq \cdots$ of cofinite subgroups Γ_n in G such that

$$\bigcap_{n=1}^{\infty} \bigcap_{g \in G} g\Gamma_n g^{-1} = \{1_G\}. \tag{3-1}$$

Consider the natural inverse sequence of homogeneous G-spaces and G-equivariant mappings intertwining them:

$$G/\Gamma_1 \longleftarrow G/\Gamma_2 \longleftarrow \cdots . \tag{3-2}$$

Denote by Y the projective limit of this sequence. A point of Y is a sequence $(g_n\Gamma_n)_{n=1}^{\infty}$ such that $g_n\Gamma_n = g_{n+1}\Gamma_n$, i.e. $g_n^{-1}g_{n+1} \in \Gamma_n$ for each $n > 0$. Endow Y with the topology of projective limit. Then Y is a compact Cantor G-space. Of course, the G-action on Y is minimal and uniquely ergodic. Denote this action by $O = (O_g)_{g \in G}$. It follows from (3-1) that O is *faithful*, i.e. $O_g \neq I$ if $g \neq 1_G$. We note that a faithful action is not necessarily free.

Definition 3.1. The dynamical system (Y, O) is called *the topological G-odometer associated with* $(\Gamma_n)_{n=1}^{\infty}$. If ν is a non-atomic Borel measure on Y which is quasi-invariant and ergodic under O then we call the dynamical system (Y, ν, O) a *nonsingular G-odometer*. By the *Haar measure* for (Y, O) we mean the unique G-invariant probability on Y.

We note that (3-1) is in no way restrictive. Indeed, let $\bigcap_{n=1}^{\infty}\bigcap_{g\in G} g\Gamma_n g^{-1} = N \neq \{1_G\}$. Define (Y,O) as above. Then N is a proper normal subgroup of G and $N = \{g \in G \mid O_g = I\}$. We now let $\widetilde{G} := G/N$ and $\widetilde{\Gamma}_n := \Gamma_n/N$. Then $\widetilde{\Gamma}_n$ is a cofinite subgroup in $\widetilde{G}$ for each $n \in \mathbb{N}$, $\widetilde{\Gamma}_1 \supsetneq \widetilde{\Gamma}_2 \supsetneq \cdots$ and $\bigcap_{n=1}^{\infty}\bigcap_{g\in\widetilde{G}} g\widetilde{\Gamma}_n g^{-1} = \{1_{\widetilde{G}}\}$. Let $(\widetilde{Y},\widetilde{O})$ denote the topological $\widetilde{G}$-odometer associated with the sequence $(\widetilde{\Gamma}_n)_{n=1}^{\infty}$. Then, of course, $Y = \widetilde{Y}$ and $O_g = \widetilde{O}_{gN}$ for each $g \in G$.

We now show that the classical nonsingular $\mathbb{Z}$-odometers of product type are covered by Definition 3.1.

Example 3.2. Let $G = \mathbb{Z}$ and let $(a_n)_{n=1}^{\infty}$ be a sequence of integers such that $a_n > 1$ for each $n \in \mathbb{N}$. We set $\Gamma_n := a_1 \cdots a_n\mathbb{Z}$. Then $\Gamma_1 \supsetneq \Gamma_2 \supsetneq \cdots$ and $\bigcap_{n=1}^{\infty}\Gamma_n = \{0\}$. The set $D_n := a_1 \cdots a_{n-1} \cdot \{0,1,\dots,a_n-1\}$ is a Γ_n-cross-section in Γ_{n-1}. Hence, in view of (3-4), the space Y of the $\mathbb{Z}$-odometer $O = (O_n)_{n\in\mathbb{Z}}$ associated with $(\Gamma_n)_{n=1}^{\infty}$ is homeomorphic to the infinite product $D = D_1 \times D_2 \times \cdots$. We identify D_n naturally with the set $\{0,1,\dots,a_n-1\}$. Then

$$D = \{0,1,\dots,a_1-1\} \times \{0,1,\dots,a_2-1\} \times \cdots .$$

To define O explicitly on this space we take $y = (y_n)_{n=1}^{\infty} \in D$. It is a routine to check that if there is $k > 0$ such that $y_j = a_j - 1$ for each $j < k$ and $y_k \neq a_k - 1$ then

$$O_1 y = (0,\dots,0,y_k+1,y_{k+1},y_{k+2},\dots).$$

If such a k does not exist, i.e. $y_j = a_j - 1$ for each $j > 0$, then $O_1 y = (0,0,\dots)$. Let κ_n be a non-degenerated probability measure on $\{0,1,\dots,a_n-1\}$ and let $\prod_{n>0}\max_{0\le d<a_n}\kappa_n(d) = 0$. This means that (i) and (ii) of Proposition 3.2 hold. Of course, Proposition 3.2(iii) holds also. Hence, by Proposition 3.2, the

nonsingular odometer

$$\left(\bigotimes_{n=1}^{\infty}\{0,\dots,a_n-1\},\bigotimes_{n=1}^{\infty}\kappa_n,O\right)$$

is of rank one. Thus, in this case our definition of nonsingular odometer coincides with the classical definition of nonsingular $\mathbb{Z}$-odometers of product type (see [1], [7]). Moreover, the $\mathbb{Z}$-odometers of product type are of rank one.

We note that there exist nonsingular (even probability preserving) G-odometers which are not of rank one.

Розділ 4
Odometer factors of nonsingular (C, F)-actions

The following concept is an “infinite” analogue of Definition 2.1.

Definition 4.1. Let a sequence $\mathcal{T} = (C_n, F_{n-1}, \kappa_n, \nu_{n-1})_{n=1}^{\infty}$ satisfy (1-1)–(1-3) and Proposition 1.8(ii) and let a sequence $(\Gamma_n)_{n=1}^{\infty}$ satisfy (3-1.) Denote by (Y, O) the topological odometer associated with $(\Gamma_n)_{n=1}^{\infty}$. Given $y = (g_n\Gamma_n)_{n=1}^{\infty} \in Y$, we say that $\mathcal{T}$ is *compatible with* y if

$$\sum_{n=1}^{\infty} \kappa_n\big(\{c \in C_n \mid c \notin g_n\Gamma_n g_n^{-1}\}\big) < \infty.$$

Denote by T the (C, F)-action of G associated with $\mathcal{T}$. Let X be the space of T and let μ stand for the the nonsingular (C, F)-measure on X determined by $(\kappa_n)_{n=1}^{\infty}$ and $(\nu_n)_{n=0}^{\infty}$. As

$$g_n\Gamma_n g_n^{-1} = g_{n+1}\Gamma_n g_{n+1}^{-1} \supset g_{n+1}\Gamma_{n+1} g_{n+1}^{-1} \qquad \text{for each } n \in \mathbb{N},$$

it follows that if $\mathcal{T}$ is compatible with y then $\mathcal{T}$ is compatible with the coset $g_n\Gamma_n \in G/\Gamma_n$ in the sense of Definition 2.1 for each $n \in \mathbb{N}$. Hence the $(\mathcal{T}, g_n\Gamma_n)$-factor mapping $\pi_{(\mathcal{T}, g_n\Gamma_n)} : X \to G/\Gamma_n$ for T is well defined (mod 0) for each $n \in \mathbb{N}$. Moreover, a measurable mapping

$$\pi_{(\mathcal{T},y)} := X \ni x \mapsto \Big(\pi_{(\mathcal{T}, g_n\Gamma_n)}(x)\Big)_{n=1}^{\infty} \in Y$$

is well defined (mod 0) too. Of course, $\pi_{(\mathcal{T},y)} \circ T_g = O_g \circ \pi_{(\mathcal{T},y)}$ for each $g \in G$. Hence the nonsingular odometer $\Big(Y, \mu \circ \pi_{(\mathcal{T},y)}^{-1}, O\Big)$ is a factor of (X, μ, T).

Definition 4.2. We call $\pi_{(\mathcal{T},y)}$ *the* $(\mathcal{T},y)$*-factor mapping* for T.

In the proposition below we find necessary and sufficient conditions (in terms of the parameters $\mathcal{T}$) under which $\pi_{(\mathcal{T},y)}$ is one-to-one, i.e. the dynamical systems (X,μ,T) and $(Y,\mu\circ\pi^{-1}_{(\mathcal{T},y)},O)$ are isomorphic via $\pi_{(\mathcal{T}T,y)}$.

Proposition 4.3. *Let $\mathcal{T}$ be compatible with y. The following are equivalent:*

(i) $\pi_{(\mathcal{T},y)}$ *is one-to-one (mod 0),*

(ii) for each $n>0$ and $\epsilon>0$ there are $l>0$ and a subset $D_l\subset G/\Gamma_l$ such that $\mu\Big([1_G]_n\triangle\pi^{-1}_{(\mathcal{T},g_l\Gamma_l)}(D_l)\Big)<\epsilon$ and

(iii) for each $n>0$ and $\epsilon>0$ there are $l>0$, a subset $D_l\subset G/\Gamma_l$ and $M>0$ such that $\nu_m\Big(C_{n+1}\cdots C_m\triangle\{f\in F_m\mid fg_l\Gamma_l\in D_l\}\Big)<\epsilon$ for each $m>M$.

Proof. (i)$\Leftrightarrow$(ii) Denote by $\mathfrak{B}$ and $\mathfrak{Y}$ the Borel σ-algebra on X and Y respectively. Of course, $\pi_{(\mathcal{T},y)}$ is one-to-one (mod 0) if and only if $\mathfrak{B}=\pi^{-1}_{(\mathcal{T},y)}(\mathfrak{Y})$ (mod 0). Since

— $\mathfrak{B}$ is generated by the family of all cylinders in X and

— $\mathfrak{B}$ is invariant under T,

it follows that $\mathfrak{B}=\pi^{-1}_{(\mathcal{T},y)}(\mathfrak{Y})$ if and only if $[1_G]_n\in\pi^{-1}_{(\mathcal{T},y)}(\mathfrak{Y})$ for each $n>0$. Let $\mathfrak{Y}_l\subset\mathfrak{Y}$ denote the finite sub-σ-algebra of subsets that are measurable with respect to the canonical projection $G\to G/\Gamma_l$. Then $\mathfrak{Y}_1\subset\mathfrak{Y}_2\subset\cdots$ and the union $\bigcup_{l=1}^{\infty}\mathfrak{Y}_l$ is dense in $\mathfrak{Y}$. It follows that $[1_G]_n\in\pi^{-1}_{(\mathcal{T},y)}(\mathfrak{Y})$ if and only if

$$\min_{A\in\mathfrak{Y}_l}\mu\Big([1_G]_n\triangle\pi^{-1}_{(\mathcal{T},y)}(A)\Big)\to 0\quad\text{as } l\to\infty.$$

It remains to note that $\{\pi^{-1}_{(\mathcal{T},y)}(A)\mid A\in\mathfrak{Y}_l\}=\{\pi^{-1}_{(\mathcal{T},g_l\Gamma_l)}(D)\mid D\subset G/\Gamma_l\}$.

(ii)$\Leftrightarrow$(iii) We note that for all $n > 0$, $l > 0$ and a subset $D \subset G/\Gamma_l$,

$$\mu\Big([1_G]_n \triangle \pi^{-1}_{(\mathcal{T}, g_l\Gamma_l)}(D)\Big) = \lim_{m\to\infty} \mu\Big([1_G]_n \triangle \pi^{-1}_{(\mathcal{T}, g_l\Gamma_l)}(D) \cap [F_m]_m\Big) \quad \text{and}$$

$$\lim_{m\to\infty} \bigotimes_{j\geq m} \kappa_j\Big(\big\{x = (c_j)_{j\geq m} \in C_m \times C_{m+1} \times \cdots \mid c_j \in g_l\Gamma_l g_l^{-1}\ \forall j \geq m\big\}\Big) = 1.$$

The latter follows from the fact that $\mathcal{T}$ is compatible with y. It implies that

$$\lim_{m\to\infty} \mu\Big(\{x = (f_m, c_{n+1}, \dots) \in [F_m]_m \mid \pi_{(\mathcal{T}, g_l\Gamma_l)}(x) = f_m g_l \Gamma_l\}\Big) = \mu([F_m]_m).$$

Hence, for each $D \subset \Gamma_l$,

$$\begin{aligned} \mu\Big([1_G]_n \triangle \pi^{-1}_{(\mathcal{T}, g_l\Gamma_l)}(D)\Big) &= \lim_{m\to\infty} \mu\left([1_G]_n \triangle \bigsqcup_{f\in F_m, f g_l\Gamma_l \in D} [f]_m\right) \\ &= \lim_{m\to\infty} \nu_m\Big(C_{n+1}\cdots C_m \triangle \big\{f \in F_m \mid f g_l \Gamma_l \in D\big\}\Big). \end{aligned}$$

This equality implies the equivalence of (ii) and (iii). $\square$

The following theorem is the main result of this section.

Theorem 4.4. *Let a sequence $\mathcal{T} = (C_n, F_{n-1}, \kappa_n, \nu_{n-1})_{n=1}^\infty$ satisfy* (1-1)–(1-3) *and Proposition 1.8(ii). Let T be the nonsingular (C,F)-action of G associated with $\mathcal{T}$ and let (Y, O) be the topological G-odometer associated with a sequence $(\Gamma_n)_{n=1}^\infty$ satisfying* (3-1). *Then for each G-equivariant measurable mapping $\tau : X \to Y$ there exist an increasing sequence $\boldsymbol{q}$ of nonnegative integers and an element $y \in Y$ such that the $\boldsymbol{q}$-telescoping $\widetilde{\mathcal{T}} = (\widetilde{C}_n, \widetilde{F}_{n-1}, \widetilde{\kappa}_n, \widetilde{\nu}_{n-1})_{n=1}^\infty$ of $\mathcal{T}$ is compatible with y and*

$$\pi_{(\widetilde{\mathcal{T}}, y)} \circ \iota_{\boldsymbol{q}} = \tau. \tag{4-1}$$

Moreover, τ is one-to-one (mod 0) if and only if for each $n > 0$ and $\epsilon > 0$ there *are* $l > 0$, *a subset* $D_l \subset G/\Gamma_l$ *and* $M > n$ *such that for each* $m > M$,

$$\widetilde{\nu}_m\Big(\widetilde{C}_{n+1}\cdots\widetilde{C}_m \triangle \{f \in \widetilde{F}_m \mid f g_l \Gamma_l \in D_l\}\Big) < \epsilon.$$

We preface the proof of Theorem 4.4 with auxiliary simple but useful facts about factor mappings.

Lemma 4.5. *Let* Γ, Γ_1 *be two cofinite subgroups in* G *and* $\Gamma_1 \subset \Gamma$. *Then*

(i) if $\mathcal{T}$ *is compatible with a coset* $g\Gamma \in G/\Gamma$ *then for each increasing sequence* $\boldsymbol{a} = (a_n)_{n=0}^{\infty}$ *of nonnegative integers with* $a_0 = 0$, *the* $\boldsymbol{a}$*-telescoping* $\widetilde{\mathcal{T}}$ *of* $\mathcal{T}$ *is also compatible with* $g\Gamma$ *and*

$$\pi_{(\mathcal{T}T, g\Gamma)} = \pi_{(\widetilde{\mathcal{T}}, g\Gamma)} \circ \iota_{\boldsymbol{a}}, \tag{4-2}$$

(ii) if $\mathcal{T}$ *is compatible with two cosets* $g\Gamma \in G/\Gamma$ *and* $g_1\Gamma_1 \in G/\Gamma_1$ *then*

$$r \circ \pi_{(\mathcal{T}, g_1\Gamma_1)} = \pi_{(\mathcal{T}, g\Gamma)} \quad \textit{if and only if} \quad g_1 g^{-1} \in \Gamma,$$

where $r : G/\Gamma_1 \to G/\Gamma$ *denotes the natural projection.*

Proof. (i) For each $n > 0$, we let $\widetilde{C}_n := C_{a_{n-1}+1}\cdots C_{a_n}$ and $\widetilde{\kappa}_n := \kappa_{a_{n-1}} * \cdots * \kappa_{a_n}$. Then

$$\widetilde{\kappa}_n\big(\{c \in \widetilde{C}_n \mid c \notin g\Gamma g^{-1}\}\big) \le \sum_{j=a_{n-1}+1}^{a_n} \kappa_j\big(\{c \in C_j \mid c \notin g\Gamma g^{-1}\}\big).$$

Hence,

$$\sum_{n=1}^{\infty} \widetilde{\kappa}_n\big(\{c \in \widetilde{C}_n \mid c \notin g\Gamma g^{-1}\}\big) \le \sum_{j=1}^{\infty} \kappa_j\big(\{c \in C_j \mid c \notin g\Gamma g^{-1}\}\big) < \infty.$$

Thus, $\widetilde{\mathcal{T}}$ is compatible with $g\Gamma$.

The formula (4-2) and the claim (ii) are verified straightforwardly. $\square$

Proof of Theorem 4.4. We note that $\tau(x) = (\tau_n(x))_{n=1}^{\infty}$ for each $x \in X$, where $\tau_n : X \to G/\Gamma_n$ is a G-equivariant mapping for every n.

From now on, we will argue inductively. At the first step we apply Remark 2.4 to T and τ_1: there exist a coset $g_1\Gamma_1 \in G/\Gamma_1$ and an increasing sequence $\boldsymbol{q}^1$ of nonnegative integers such that the $\boldsymbol{q}^1$-telescoping $\mathcal{T}_1$ of $\mathcal{T}$ is $g_1\Gamma_1$-compatible and

$$\pi_{(\mathcal{T}_1, g_1\Gamma_1)} \circ \iota_{\boldsymbol{q}^1} = \tau_1. \tag{4-3}$$

At the second step we we apply Remark 2.4 to the (C, F)-action $\widetilde{T}$ of G associated with $\mathcal{T}_1$ and the factor mapping $\tau_2 \circ \iota_{\boldsymbol{q}^1}{}^{-1}$ of $\widetilde{T}$: there exist a coset $g_2\Gamma_2 \in G/\Gamma_2$ and and an increasing sequence $\boldsymbol{q}^2$ of nonnegative integers such that the $\boldsymbol{q}^2$-telescoping $\mathcal{T}_2$ of $\mathcal{T}_1$ is $g_2\Gamma_2$-compatible and

$$\pi_{(\mathcal{T}_2, g_2\Gamma_2)} \circ \iota_{\boldsymbol{q}^2} = \tau_2 \circ \iota_{\boldsymbol{q}^1}{}^{-1}. \tag{4-4}$$

Consider the natural projection $\omega_{2,1} : G/\Gamma_2 \to G/\Gamma_1$. Since $\omega_{2,1} \circ \tau_2 = \tau_1$, it follows from (4-3) and (4-4) that

$$\omega_{2,1} \circ \pi_{(\mathcal{T}_2, g_2\Gamma_2)} \circ \iota_{\boldsymbol{q}^2} = \pi_{(\mathcal{T}_1, g_1\Gamma_1)}$$

Then Lemma 4.5(i),(ii) imply that $g_2 g_1^{-1} \in \Gamma_1$. Continuing inductively, we obtain a sequence $(g_n)_{n=1}^{\infty}$ of elements in G and a sequence $(\boldsymbol{q}^n)_{n=1}^{\infty}$ of increasing sequences of nonnegative integers such that for each $n > 0$,

(α_1) $g_n g_{n-1}^{-1} \in \Gamma_{n-1}$,

(α_2) the $\boldsymbol{q}^n$-telescoping $\mathcal{T}_n = (C_k^{(n)}, F_{k-1}^{(n)}, \kappa_k^{(n)}, \nu_{k-1}^{(n)})_{k=1}^{\infty}$ of $\mathcal{T}_{n-1}$ is compatible with the coset $g_n\Gamma_n$ and

(α_3) $\pi_{(\mathcal{T}_n, g_n\Gamma_n)} \circ \iota_{\boldsymbol{q}^1 \circ \cdots \circ \boldsymbol{q}^n} = \tau_n$.

We now choose an integer $a_n > 0$ large so that

(α_4) $\displaystyle\sum_{k\geq a_n} \kappa_k^{(n)}(\{c \in C_k^{(n)} \mid c \notin g_n\Gamma_n g_n^{-1}\}) < \frac{1}{n^2}$

for each $n > 0$. It follows from (α_1) that the sequence $y := (g_n\Gamma_n)_{n=1}^{\infty}$ is a well-defined element of Y. Of course,

$\boldsymbol{q}^1 \circ \boldsymbol{q}^2$ is a subsequence of $\boldsymbol{q}^1$,

$\boldsymbol{q}^1 \circ \boldsymbol{q}^2 \circ \boldsymbol{q}^3$ is a subsequence of $\boldsymbol{q}^1 \circ \boldsymbol{q}^2$

and so on. Hence, utilizing the diagonalization method, we can construct a sequence $\boldsymbol{q} = (q_n)_{n=1}^{\infty}$ of integers such that

— $0 = q_0 < q_1 < \cdots < q_n$,

— $\boldsymbol{q}$ is a subsequence of $\boldsymbol{q}^1 \circ \cdots \circ \boldsymbol{q}^n$ and

— $q_n \geq a_n$

for every $n > 0$. Denote by $\widetilde{\mathcal{T}}$ the $\boldsymbol{q}$-telescoping $\widetilde{\mathcal{T}} = (\widetilde{C}_n, \widetilde{F}_{n-1}, \widetilde{\kappa}_n, \widetilde{\nu}_{n-1})_{n=1}^{\infty}$ of $\mathcal{T}$. We are going to show that $\widetilde{\mathcal{T}}$ is compatible with y. By the construction of $\boldsymbol{q}$, for each $n > 0$, there are integers $d_{n,2} \geq d_{n,1} \geq a_n$ such that

$$\widetilde{C}_n = C_{d_{n,1}}^{(n)} \cdots C_{d_{n,2}}^{(n)} \qquad \text{and} \qquad \widetilde{\kappa}_n = \kappa_{d_{n,1}} * \cdots * \kappa_{d_{n,2}}.$$

Therefore we deduce from (α_4) that

$$\widetilde{\kappa}_n(\{c \in \widetilde{C}_n \mid c \notin g_n\Gamma_n g_n^{-1}\}) \leq \sum_{k=d_{n,1}}^{d_{n,2}} \kappa_k^{(n)}(\{c \in C_k^{(n)} \mid c \notin g_n\Gamma_n g_n^{-1}\}) < \frac{1}{n^2}.$$

Hence, $\displaystyle\sum_{n=1}^{\infty} \widetilde{\kappa}_n(\{c \in \widetilde{C}_n \mid c \notin g_n\Gamma_n g_n^{-1}\}) < \infty$, i.e. $\widetilde{\mathcal{T}}$ is compatible with y, as desired.

We now prove (4-1). Of course, $\widetilde{\mathcal{T}}$ is a telescoping of the $\boldsymbol{q}^1 \circ \cdots \circ \boldsymbol{q}^n$-telescoping of $\mathcal{T}$ for each n. In view of (α_2), the $\boldsymbol{q}^1 \circ \cdots \circ \boldsymbol{q}^n$-telescoping of $\mathcal{T}$

equals $\mathcal{T}_n$. Thus, $\widetilde{\mathcal{T}}$ is a telescoping of $\mathcal{T}_n$. Denote by θ_n the canonical isomorphism corresponding to this telescoping. Then $\iota_{\boldsymbol{q}} = \theta_n \circ \iota_{\boldsymbol{q}^1\circ\cdots\circ\boldsymbol{q}^n}$. It follows from this and (α_3) that

$$\pi_{(\widetilde{\mathcal{T}},g_n\Gamma_n)} \circ \iota_{\boldsymbol{q}} = \pi_{(\widetilde{\mathcal{T}},g_n\Gamma_n)} \circ \theta_n \circ \iota_{\boldsymbol{q}^1\circ\cdots\circ\boldsymbol{q}^n} = \pi_{(\mathcal{T}_n,g_n\Gamma_n)} \circ \iota_{\boldsymbol{q}^1\circ\cdots\circ\boldsymbol{q}^n} = \tau_n$$

for each $n \in \mathbb{N}$. Hence $\pi_{(\widetilde{\mathcal{T}},y)} = \tau$, as desired.

The second (the last) claim of the theorem follows from the first one and Proposition 4.3. $\square$

As a corollary, we obtain a criterion for the existence (or non-existence) of odometer factors for rank-one nonsingular actions.

Corollary 4.6. *Let T be the nonsingular (C, F)-action of G associated with a sequence $\mathcal{T} = (C_n, F_{n-1}, \kappa_n, \nu_{n-1})_{n=1}^\infty$ satisfying* (1-1)–(1-3) *and Proposition 1.8(ii). Then T has no nonsingular odometer factors if and only if for each decreasing sequence $\Gamma_1 \supsetneq \Gamma_2 \supsetneq \cdots$ of cofinite subgroups in G satisfying* (3-1), *no telescoping of $\mathcal{T}$ is compatible with the sequence $(\Gamma_n)_{n=1}^\infty$, i.e. for each sequence $0 = q_1 < q_2 < \cdots$,*

$$\sum_{n=0}^{\infty} \kappa_{q_n+1} * \cdots * \kappa_{q_{n+1}}(\{c \in C_{q_n+1} \cdots C_{q_{n+1}} \mid c \notin \Gamma_n\}) = \infty.$$

Proof. It is sufficient to utilize Theorem 4.4 and the following remark.

Let Y be a G-odometer associated with a decreasing sequence $(\Gamma_n)_{n=1}^\infty$ of cofinite subgroups in G such that (3-1) holds. Let $y \in Y$. Then $y = (g_n\Gamma_n)_{n=1}^\infty$ with $g_n g_{n+1}^{-1} \in \Gamma_n$ for each $n \in \mathbb{N}$. Of course, $g_1\Gamma_1 g_1^{-1} \supset g_2\Gamma_2 g_2^{-1} \supset \cdots$ and the sequence $(g_n\Gamma_n g_n^{-1})_{n=1}^\infty$ satisfies (3-1). Denote by Y_y the space of the G-odometer associated with $(g_n\Gamma_n g_n^{-1})_{n=1}^\infty$. Then there is a canonical G-equivariant homeomorphism $\varphi_y : Y \to Y_y$. It is well defined by the formula

$$\varphi_y\big((z_n\Gamma_n)_{n=1}^\infty\big) := (z_n\Gamma_n g_n^{-1})_{n=1}^\infty = \big(z_n g_n^{-1}(g_n\Gamma_n g_n^{-1})\big)_{n=1}^\infty.$$

It follows that there is a G-equivariant map from X to Y if and only if there is G-equivariant map from X to Y_y. $\square$

In a similar way, we obtain a criterion when a nonsingular (C,F)-action is not isomorphic to any nonsingular odometer.

Corollary 4.7. *Let T be the nonsingular (C,F)-action of G associated with a sequence $\mathcal{T}$ satisfying* (1-1)–(1-3) *and Proposition 1.8(ii). Then T is not isomorphic to any nonsingular odometer if and only if for each decreasing sequence $\Gamma_1 \supsetneq \Gamma_2 \supsetneq \cdots$ of cofinite subgroups in G satisfying* (3-1) *and each increasing sequence $\boldsymbol{q}$ of nonnegative integers such that the $\boldsymbol{q}$-telescoping $\widetilde{\mathcal{T}}$ of $\mathcal{T}$ is compatible with the point $(\Gamma_n)_{n=1}^\infty \in \operatorname{proj}\lim_{n\to\infty} G/\Gamma_n$, there exist $n>0$ and $\epsilon_0>0$ such that for each $l>0$, $D_l \subset G/\Gamma_l$ and $M>n$, there is $m>M$ with*

$$\widetilde{\nu}_m\Big(\widetilde{C}_{n+1}\cdots\widetilde{C}_m \triangle \{f\in \widetilde{F}_m \mid fg_l\Gamma_l \in D_l\}\Big) > \epsilon_0.$$

Розділ 5
Example of Heisenberg group action of rank one

In this section we consider the 3-dimensional discrete Heisenberg group $H_3(\mathbb{Z})$. We remind that

$$H_3(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x, y, z \in \mathbb{Z} \right\}.$$

This group is non-Abelian, nilpotant and residually finite. For brevity, we will write (x, y, z) for the matrix $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$. It is straightforward to verify that

$$(x, y, z) \cdot (x_1, y_1, z_1) = (x + x_1, y + y_1, z + z_1 + xy_1).$$

$$(x, y, z) \cdot (x_1, y_1, z_1) \cdot (x, y, z)^{-1} = (x_1, y_1, z_1 + xy_1 - yx_1).$$

The center of $H_3(\mathbb{Z})$ is $\{(0, 0, z) \mid z \in \mathbb{Z}\}$. Given $a, b, c \geq 0$, we let

$$\Pi(a, b, c) := \{(x, y, z) \in H_3(\mathbb{Z}) \mid 0 \leq x < a, 0 \leq y < b, 0 \leq z < c\}.$$

It is straightforward to verify that if $a_n \to +\infty$, $b_n \to +\infty$, $c_n \to +\infty$ and $\dfrac{b_n}{c_n} \to 0$ as $n \to \infty$ then $(\Pi(a_n, b_n, c_n))_{n=1}^{\infty}$ is a left Følner sequence in $H_3(\mathbb{Z})$. If, in addition, $\dfrac{a_n}{c_n} \to 0$ then $(\Pi(a_n, b_n, c_n))_{n=1}^{\infty}$ is a 2-sided Følner sequence in $H_3(\mathbb{Z})$.

Example 5.1. We construct a probability preserving (C, F)-action T of

$H_3(\mathbb{Z})$ that has an odometer factor but T itself is not isomorphic to any odometer. For that, we first define recurrently a sequence $(h_n)_{n=0}^\infty$ of positive integers by setting

$$h_0 := 1 \text{ and } h_{n+1} := 16h_n + 9 \cdot 4^{n+1}.$$

It is easy to check that $h_n = 4^{2n+1} - 3 \cdot 4^n$ for each $n \geq 0$. We set

$$C_{n+1}^{(1)} := \big\{(a, b, 0) \mid a, b \in \{0, 2^n\}\big\} \text{ and}$$
$$C_{n+1}^{(2)} := \{(0, 0, jh_n) \mid j = 0, \dots, 7\} \cdot \{(0, 0, 0), (0, 0, 8h_n + 2 \cdot 4^{n+1})\}.$$

We now define a sequence $(C_n, F_{n-1})_{n=1}^\infty$ by setting

$$F_n := \Pi(2^n, 2^n, h_n) \text{ and } C_{n+1} := C_{n+1}^{(1)} C_{n+1}^{(2)} \qquad \text{for each } n \geq 0.$$

It is straightforward to check that (1-1) is satisfied. We define measures κ_n on C_n and ν_n on F_n by setting $\nu_0(1) = 1$ and

$$\kappa_n(c) := \frac{1}{\#C_n} = \frac{1}{64}, \quad \nu_n(f) := \frac{1}{64^n} \qquad \text{for each } c \in C_n,\ f \in F_n,\ n > 0.$$

Let $\mathcal{T} := (C_n, F_{n-1}, \kappa_n, \nu_{n-1})$. Then (1-2) and (1-3) hold for $\mathcal{T}$. Since $\#F_n = 4^n(4^{2n+1} - 3 \cdot 4^n)$ and $\#C_{n+1} = 64$, we obtain that

$$\prod_{n>0} \frac{\#F_{n+1}}{\#F_n \#C_{n+1}} = \prod_{n>0} \frac{4^{n+2} - 3}{4(4^{n+1} - 3)} = \prod_{n>0} \left(1 + \frac{9}{4(4^{n+1} - 3)}\right) < \infty. \tag{5-1}$$

Of course, $(F_n)_{n=1}^\infty$ is a 2-sided Følner sequence in $H_3(\mathbb{Z})$. Hence, Proposition 1.8(ii) holds. Then the (C, F)-action $T = (T_g)_{g \in H_3(\mathbb{Z})}$ associated with $\mathcal{T}$ is well defined on a measure space (X, μ), where μ is the (C, F)-measure determined by $(\kappa_n, \nu_{n-1})_{n=1}^\infty$. Moreover, T preserves μ, i.e. μ is the Haar measure for the (C, F)-equivalence relation on X, and $\mu(X) < \infty$ in view of (5-1) (see Remark 1.5).

Next, we define a measure preserving odometer action of $H_3(\mathbb{Z})$. Given $n > 0$, we let

$$\Gamma_n := \{(i \cdot 2^n, j \cdot 2^n, k \cdot 2^n) \in H_3(\mathbb{Z}) \mid i, j, k \in \mathbb{Z}\}.$$

Then Γ_n is a normal cofinite subgroup of $H_3(\mathbb{Z})$, $\Gamma_1 \supsetneq \Gamma_2 \supsetneq \cdots$ and $\bigcap_{n=1}^{\infty} \Gamma_n = \{1\}$. Denote by O the $H_3(\mathbb{Z})$-odometer associated with the sequence $(\Gamma_n)_{n=1}^{\infty}$. We call it *the 2-adic odometer* action of $H_3(\mathbb{Z})$. This odometer is normal. It is defined on the compact metric group

$$Y := \operatorname*{proj\,lim}_{n\to\infty} H_3(\mathbb{Z})/\Gamma_n.$$

Denote by ν the Haar measure on Y. Since $C_{n+1} \subset \Gamma_n$ for each $n \in \mathbb{N}$, it follows that $\mathcal{T}$ is compatible with $(\Gamma_n)_{n=1}^{\infty} \in Y$. Hence the $(\mathcal{T}, (\Gamma_n)_{n=1}^{\infty})$-factor mapping $\pi_{(\mathcal{T},(\Gamma_n)_{n=1}^{\infty})} : X \to Y$ intertwines T with O. For brevity, instead of $\pi_{(\mathcal{T},(\Gamma_n)_{n=1}^{\infty})}$ below we will write π. The measure $\mu \circ \pi^{-1}$ on Y is finite and invariant under O. Since O is uniquely ergodic, it follows that $\mu \circ \pi^{-1} = \mu(X) \cdot \nu$. Thus, the 2-adic $H_3(\mathbb{Z})$-odometer is a finite measure preserving factor of (X, μ, T).

We claim that (Y, ν, O) is the *maximal odometer factor* of T, i.e. every odometer factor of T is a factor of O. Let $\Lambda_1 \supset \Lambda_2 \supset \cdots$ be a sequence of cofinite subgroups in $H_3(\mathbb{Z})$ with $\bigcap_{n=1}^{\infty} \bigcap_{g\in G} g\Lambda_n g^{-1} = \{1\}$ and let $Q = (Q_g)_{g\in H_3(\mathbb{Z})}$ stand for the associated $H_3(\mathbb{Z})$-odometer which is defined on the space $Z := \operatorname*{proj\,lim}_{n\to\infty} H_3(\mathbb{Z})/\Lambda_n$ equipped with Haar measure. If there is an $H_3(\mathbb{Z})$-equivariant mapping $\tau : X \to Z$ then, by Theorem 4.4, there exist a sequence $\boldsymbol{q} = (q_n)_{n=0}^{\infty}$ and an element $z = (g_n\Lambda_n)_{n=1}^{\infty} \in Z$ such that $0 = q_0 < q_1 < q_2 < \cdots$, the $\boldsymbol{q}$-telescoping $\widetilde{\mathcal{T}}$ of $\mathcal{T}$ is compatible with z and $\tau = \pi_{(\widetilde{\mathcal{T}},z)} \circ \iota_{\boldsymbol{q}}$. Replacing Λ_n with $g_n^{-1}\Lambda_n g_n$ for each $n \in \mathbb{N}$, we pass to an isomorphic (to Q) odometer as a factor of X (see the proof of Corollary 4.6). We denote it by the same symbol Z. Therefore, without loss of generality, we may assume that

$\widetilde{\mathcal{T}}$ is compatible with $z = (\Lambda_n)_{n=1}^{\infty}$. It follows that there is $N > 0$ such that for each $n > N$

$$\#\{c \in C_{q_n+1} \cdots C_{q_{n+1}} \mid c \notin \Lambda_n\} < 64^{-1}.$$

As $\#C_{q_n+1} = 64$ and $\#(C_{q_n+1} \cdots C_{q_{n+1}}) = \#(C_{q_n+1})\#(C_{q_n+2} \cdots C_{q_{n+1}})$, a version of Fubuni theorem yields that there exists at least one element $d \in C_{q_n+2} \cdots C_{q_{n+1}}$ such that $C_{q_n+1}d \subset \Lambda_n$. Hence,

$$\{(2^{q_n}, 0, 0), (0, 2^{q_n}, 0)\} \subset \{\widetilde{c}c^{-1} \mid \widetilde{c}, c \in C_{q_n+1}\} \subset \Lambda_n.$$

Therefore, the subgroup

$$\Sigma_n := \{(i \cdot 2^{q_n}, j \cdot 2^{q_n}, k \cdot 4^{q_n}) \in H_3(\mathbb{Z}) \mid i, j, k \in \mathbb{Z}\}$$

is contained in Λ_n. This implies that $\Gamma_{2q_n} \subset \Sigma_n \subset \Lambda_n$ for each $n > 0$. The natural projection

$$H_3(\mathbb{Z})/\Gamma_{2q_n} \to H_3(\mathbb{Z})/\Lambda_n$$

is $H_3(\mathbb{Z})$-equivariant for each n. Passing to the projective limit as $n \to \infty$, we obtain a $H_3(\mathbb{Z})$-equivariant projection $\eta : Y \to Z$. It is straightforward to verify that $\tau = \eta \circ \pi$, as desired.

It remains to show that π is not one-to-one $(\bmod\, 0)$. For that, it suffices to show that the restriction of π to X_0 is not one-to-one. Let κ_n^1 and κ_n^2 be the equidistributions on $C_n^{(1)}$ and $C_n^{(2)}$ respectively, $n \in \mathbb{N}$. We let

$$(X_0^{(1)}, \kappa^1) := \left(C_1^{(1)} \times C_2^{(1)} \times \cdots, \bigotimes_{n=1}^{\infty} \kappa_n^1\right) \quad \text{and}$$
$$(X_0^{(2)}, \kappa^2) := \left(C_1^{(2)} \times C_2^{(2)} \times \cdots, \bigotimes_{n=1}^{\infty} \kappa_n^2\right).$$

Then $X_0, X_0^{(1)}$ and $X_0^{(2)}$ are compact subsets of the Polish group $H_3(\mathbb{Z})^{\mathbb{N}}$. The mapping

$$\alpha : X_0^{(1)} \times X_0^{(2)} \ni (x^1, x^2) \mapsto x^1 x^2 \in X_0$$

is a homeomorphism that maps the product measure $\kappa^1 \otimes \kappa^2$ to $\mu \restriction X_0$. Moreover, for each $(x^1, x^2) \in X_0^{(1)} \times X_0^{(2)}$,

$$\pi(x^1 x^2) = \pi(x^1)\pi(x^2). \tag{5-2}$$

Let $\mathcal{Z}$ stand for the center of $H_3(\mathbb{Z})$. Denote by Y_0 the center of Y. It is routine to verify that

$$Y_0 = \operatorname*{proj\,lim}_{n\to\infty} \mathcal{Z}\Gamma_n/\Gamma_n = \operatorname*{proj\,lim}_{n\to\infty} \mathcal{Z}/(\mathcal{Z} \cap \Gamma_n).$$

Denote by Y_2 the quotient group Y/Y_0 and by ω the quotient homomorphism $Y \to Y_2$. Then we obtain a short exact sequence of compact totally disconnected groups

$$1 \longrightarrow Y_0 \longrightarrow Y \overset{\omega}{\longrightarrow} Y_2 \longrightarrow 1.$$

Denote by λ_0 and λ_2 the Haar measures on Y_0 and Y_2 respectively. It is straightforward to verify that

$$Y_2 = \operatorname*{proj\,lim}_{n\to\infty} H_3(\mathbb{Z})/(\mathcal{Z}\Gamma_n) = \operatorname*{proj\,lim}_{n\to\infty} \mathbb{Z}^2/2^n\mathbb{Z}^2 \quad \text{and}$$

$$\omega \circ \pi(x^1) = \big((a_1, b_1) + 2\mathbb{Z}^2, (a_2, b_2) + 2^2\mathbb{Z}^2, \dots\big)$$

for each $x^1 = \big((a_1, b_1, 0), (a_2, b_2, 0), \dots\big) \in X_0^{(1)}$. Hence $\omega \circ \pi$ is a measure preserving homeomorphism of $(X_0^{(1)}, \kappa^1)$ onto (Y_2, λ_2). We now define a continuous mapping $s : Y_2 \to Y$ by setting

$$s(\omega(\pi(x^1))) := \pi(x^1). \tag{5-3}$$

Then s is a cross-section of ω. Hence, the mapping

$$\beta : Y \ni y \mapsto (ys(\omega(y))^{-1}, \omega(y)) \in Y_0 \times Y_2$$

is a well defined measure preserving homeomorphism of (Y, ν) onto the product measure space $\big(Y_0 \times Y_2, \lambda_0 \otimes \lambda_2\big)$. It follows from (5-2) and (5-3) that

$$\beta \circ \pi \circ \alpha(x^1, x^2) = \omega(\pi(x^1)\pi(x^2)) = \big(\pi(x^2), \omega \circ \pi(x^1)\big)$$

for each $(x^1, x^2) \in X_0^{(1)} \times X_0^{(2)}$. Moreover,

$$(\kappa^1 \otimes \kappa^2) \circ (\beta \circ \pi \circ \alpha)^{-1} = \lambda_0 \otimes \lambda_2.$$

Therefore, π is not one-to-one (μ-mod 0) if and only if the mapping $\pi \restriction X_0^{(2)} \to Y_0$ is not one-to-one ($\mu^{(2)}$-mod 0).

Our purpose now is to show that $\pi \restriction X_0^{(2)} \to Y_0$ is not one-to-one. Let

$$\begin{aligned} C_{n+1}^{(3)} &:= \{0, h_n, 2h_n, 3h_n\}, \\ C_{n+1}^{(4)} &:= \{0, 4h_n\} + \{0, 8h_n + 2 \cdot 4^{n+1}\}, \end{aligned}$$

κ_n^3 and κ_n^4 are the equidistributions on $C_n^{(3)}$ and $C_n^{(4)}$ respectively for each $n \geq 0$ and let

$$\begin{aligned} (X_0^{(3)}, \kappa^3) &:= \left(C_1^{(3)} \times C_2^{(3)} \times \cdots, \bigotimes_{n=1}^{\infty} \kappa_n^3\right) \quad \text{and} \\ (X_0^{(4)}, \kappa^4) &:= \left(C_1^{(4)} \times C_2^{(4)} \times \cdots, \bigotimes_{n=1}^{\infty} \kappa_n^4\right). \end{aligned}$$

Then $X_0^{(3)}$ and $X_0^{(4)}$ are compact subsets of $H_3(\mathbb{Z})^{\mathbb{N}}$. The mapping

$$X_0^{(3)} \times X_0^{(4)} \ni (x^3, x^4) \mapsto x^3 x^4 \in X_0^{(2)}$$

is a well defined homeomorphism that maps the product measure $\kappa^3 \otimes \kappa^4$ to

κ^2. Moreover, for each $(x^3, x^4) \in X_0^{(3)} \times X_0^{(4)}$,

$$\pi(x^3 x^4) = \pi(x^3)\pi(x^4).$$

In view of that, it suffices to show that the mapping $\pi \restriction X^{(3)} \to Y_0$ is a a bijection and $\kappa^3 \circ (\pi \restriction X^{(3)})^{-1} = \lambda_0$. We leave a routine verification of these facts to the reader.

Висновки

In this work we found the explicit criterion when the action of nonsingular rank one is isomorphic to the odometer, and when the odometer is the factor of the nonsingular rank one transformation. It is a step forward to a solution of the isomorphism problem in Dynamical Systems. This work can be used for solving this problem further and for generalizations to broader classes of transformations.

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