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# Кваліфікаційна робота

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(освітньо-кваліфікаційний рівень)

на тему: Цілком геодезичні та мінімальні векторні поля на групі осцилятора


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Анотація

В цій роботі розглядається група осцилятора. Класична конструкція дозволяє зробити для цієї групи Лі алебру Лі з природною метрикою Сасакі на дотичному розшаруванні, де зручно вивчати векторні поля та їхні властивості.

Іншими авторами вже були досліджені множини одиничних лівоінваріантних гармонічних векторних полів та тих, що задають гармонічне відображення, тому в цій роботі було поставлено питання щодо мінімальних та повністю геодезичних одиничних лівоінваріантних полів.

Було знайдено всі мінімальні та всі повністю геодезичні одиничні лівоінваріантні векторні поля для випадку 4-вимірної групи осцилятора. Також було знайдено всі геодезичні векторні поля та знайдено множину мінімальних одиничних лівоінваріантних векторних полів, геодезичних та ортогональних одновимірному підпростору, натягнутому на останній, спеціальний базисний вектор у багатовимірному випадку, та сформульовано досить зручну теорему для перевірки на мінімальність одиничного векторного поля на метриці Сасакі дотичного розшарування риманового многовида.

Abstract

In this thesis we consider the oscillator group. Using classical construction of Lie algebra for this Lie group, we are allowed to build natural Sasaki metric on it's tangent bundle, where we can study vector fields and their properties.

The sets of left-invariant harmonic unit vector fields and vector fields defining harmonic maps have been studied by other authors, so in this thesis we consider the question about minimal and totally geodesic unit left-invariant vector fields.

We have found all minimal and all totally geodesic unit left-invariant vector fields in the case of 4-dimentional oscillator group. We have also found all geodesic vector fields and the set of all minimal unit left-invariant vector fields, geodesic and orthogonal to the one-dimensional subspace spanned by the last, special basis vector in the multidimensional case and we also formulate a theorem for checking the minimality of a unit vector field on the Sasaki metric on the tangent bundle of a Riemannian manifold.

## 2 Introduction

A unit vector field $\xi$ on the Riemannian manifold $(M, g)$ defines a mapping $\xi : M \to T_1M$ into a unit tangent bundle $T_1M$ . The canonic Riemannian metric $g_S$ (the Sasakian one) on the tangent bundle $TM$ gives rise to Riemannian metric on the image $\xi(M) \subset T_1M \subset TM$. In this way $\xi(M)$ endowed with the intrinsic and extrinsic geometry.

Consider 4-dimensional oscillator group G. It is connected simply connected solvable non-nilpotent Lie group whose Lie algebra $g$ is the oscillator algebra, linearly spanned by 4 elements, let's call them $E_1, E_2, E_3, E_4$. Then the only non-zero commutator relations are:

$$[E_2, E_3] = E_1,\ [E_2, E_4] = -E_3,\ [E_3, E_4] = E_2$$

It decomposes in a sum of Abelian Lie algebra, spanned by three first elements, and one dimentional Abelian Lie algebra for the fourth and may be realised as $H_3 \times \mathbb{R}$, where $H_3$ is Heisenberg group of dimention three.

In the $2n + 2$ dimentional case $G_n(\lambda) = G(\lambda_1, \lambda_2, \cdots , \lambda_n)$, and again we have a connected simply connected solvable non-nilpotent Lie group, whose Lie algebra $g_n(\lambda) = g_n(\lambda_1\lambda_2 \cdots \lambda_n)$ is the oscillator algebra, spanned by 2n+2 elements, $X_1, \cdots , X_n, Y_1, \cdots , Y_n, P, Q$, whose non-zero commutator relations are:

$$[X_i, Y_j] = \delta_{ij}P,\ [Q, X_j] = \lambda_jY_j,\ [Q, Y_j] = -\lambda_jX_j,\ 1 \le i, j \le n$$

We know it could be considered as a product of $2n+1$-dimentional subgroup (odd-dimentional Heisenberg group) by 1-dimentional subgroup, the linear span of the last, special basis vector.

In [6](2019) left-invariant harmonic unit vector fields and vector fields defining harmonic maps were obtained for $2n + 2$-dimentional oscillator group. Furthermore, in [5](2020) in a special left-invariant metric the set of all Killing, parallel and geodesic vector fields (on 4-dimentional oscillator group for that metric they are the same) and some other properties of the oscillator group were also studied.

In this thesis we study the sets of all minimal and totally geodesic unit vector field on 4-dimentional oscillator group and find a new minimality criteria for a unit vector field.

Furthermore, when our group is $G(\lambda_1, \lambda_2, \cdots , \lambda_n) = G(\lambda)$, i.e. $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \lambda$, which we consider firstly for the case $\lambda = 1$, it is proved in [6] that the set of $\left\{P, Q, \sum_{i=1}^{n} a_iX_i + a_{n+i}Y_i | \sum_{i=1}^{2n} a_i^2 = 1\right\}$ is the set of all left-invariant harmonic unit vector fields and vector fields defining harmonic maps for $2n + 2$-dimentional oscillator group, and it appears again with other properties in [5]. In 4-dimentional case (n=1) we find it is also the set of all minimal unit vector fields on G ($E_4, E_3$ or any unit vector field in the plane, spanned by $E_1, E_2$). Therefore we have decided to check those fields in $2n + 2$ dimentional case also. We have proved this set is the set of all geodesic vector fields on G. Then we consider apart, as they make direct product, $Q$ and its complement subspace. $Q$ is obviously geodesic, minimal and totally geodesic. $P$ is also geodesic and minimal, and we checked also $\xi \in P^\perp \subset Q^\perp$ and find out all of them are also geodesic and minimal.

**Theorem 1** *Let $G$ be Riemannian manifold, equipped with Sasaki metric on it's tangent bundle, and let $\xi = (a_1, a_2, ..a_n)$ arbitrary unit vector field. Then $\xi$ is minimal if and only if:*

$$\sum_{i=0}^{n-1} \frac{1}{1+\sigma_i^2}\Big((\nabla_{e_i} A_\xi)e_i + A_\xi R(\xi, A_\xi e_i)e_i\Big) = \Big(\sum_{i=0}^{n-1} \frac{\sigma_i^2}{1+\sigma_i^2}\Big)\xi$$

*Where $\sigma_i$ are singular values for $\xi$, i.e. square roots of eigenvalues of $A_\xi^t A_\xi$, $A_\xi X = -\nabla_X \xi$ - covariant derivation operator.*

**Theorem 2** *Let $G$ be the oscillator group, spanned by $E_1, E_2, E_3, E_4$, equipped with Sasaki metric on it's tangent bundle, and let $\xi = a_1E_1 + a_2E_2 + a_3E_3 + a_4E_4$ arbitrary unit vector field. Then $\xi$ is minimal if and only if:*

*$a_4 = 1, a_1 = a_2 = a_3 = 0$, which means $\xi = E_4$, or*

*$a_3 = 1, a_1 = a_2 = a_4 = 0$, which means $\xi = E_3$, or*

*$a_2 = a_4 = 0$, which means $\xi = a_1E_1 + a_2E_2$, $a_1^2 + a_2^2 = 1$.*

**Theorem 3** *Let $G$ be the oscillator group, spanned by $E_1, E_2, E_3, E_4$, equipped with Sasaki metric on it's tangent bundle, then the only totally geodesic unit vector field is $E_4$.*

**Theorem 4** *Let $G$ be 2n+2-dimentional oscillator group, spanned by*

$$X_1, \cdots, X_n, Y_1, \cdots, Y_n, P, Q$$

*equipped with Sasaki metric on it's tangent bundle, and let*

$$\xi = \sum_{i=1}^{n}(a_iX_i + a_{n+i}Y_i) + a_{2n+1}P + a_{2n+2}Q$$

*arbitrary unit vector field.*

*Then if $\xi$ is either $P$ or $Q$ or $\sum_{i=1}^{n}(a_iX_i + a_{n+i}Y_i)$, $\sum_{i=1}^{2n} a_i^2 = 1$ it is gedesic and minimal. There are no other geodesic unit vector fields on $G$.*

# 3 Preliminaries

Consider 4-dimentional oscillator group G.

Let's call standart basis $E_1, E_2, E_3, E_4$. The only non-zero commutator relations are:

$$[E_2, E_3] = E_1, [E_2, E_4] = -E_3, [E_3, E_4] = E_2$$

Define $\xi$ arbitrary unit vector field $\xi = (a_1, a_2, a_3, a_4)$, operator $A_\xi X = -\nabla_X \xi$.

Applying it to the standart basis, we obtain matrix of this linear operator:

$$A_\xi = \begin{pmatrix} 0 & \frac{a_3}{2} & \frac{a_2}{2} & -a_2 \\ -\frac{a_3}{2} & 0 & -\frac{a_1}{2} & a_1 \\ \frac{a_2}{2} & -\frac{a_1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Eigenvalues and eigenvectors of $A_\xi$ are:

$$\lambda_{0,1} = 0, v_0 = (-a_1, -a_2, a_3, 0), v_1 = (2a_1, 2a_2, 0, a_3)$$
$$\lambda_2 = \frac{1}{2}\sqrt{a_1^2 + a_2^2 - a_3^2},$$
$$v_2 = (a_2\sqrt{a_1^2 + a_2^2 - a_3^2} - a_1a_3, -a_1\sqrt{a_1^2 + a_2^2 - a_3^2} - a_2a_3, a_1^2 + a_2^2, 0)$$
$$\lambda_3 = -\frac{1}{2}\sqrt{a_1^2 + a_2^2 - a_3^2},$$
$$v_3 = (-a_2\sqrt{a_1^2 + a_2^2 - a_3^2} - a_1a_3, a_1\sqrt{a_1^2 + a_2^2 - a_3^2} - a_2a_3, a_1^2 + a_2^2, 0)$$

Then we calculate matrix $A^tA$ and it's eigenvectors and eigenvalues in order to obtain singular basis and singular numbers.

$$A_\xi^t A_\xi = \begin{pmatrix} \frac{1}{4}\,{a_3}^2 + \frac{1}{4}\,{a_2}^2 & -\frac{1}{4}\,a_1a_2 & \frac{1}{4}\,a_1a_3 & -\frac{1}{2}\,a_1a_3 \\ -\frac{1}{4}\,a_1a_2 & \frac{1}{4}\,{a_3}^2 + \frac{1}{4}\,{a_1}^2 & \frac{1}{4}\,a_2a_3 & -\frac{1}{2}\,a_2a_3 \\ \frac{1}{4}\,a_1a_3 & \frac{1}{4}\,a_2a_3 & \frac{1}{4}\,{a_2}^2 + \frac{1}{4}\,{a_1}^2 & -\frac{1}{2}\,{a_2}^2 - \frac{1}{2}\,{a_1}^2 \\ -\frac{1}{2}\,a_1a_3 & -\frac{1}{2}\,a_2a_3 & -\frac{1}{2}\,{a_2}^2 - \frac{1}{2}\,{a_1}^2 & {a_1}^2 + {a_2}^2 \end{pmatrix}$$

We call $\sigma_i$ singular values, i.e. square roots of eigenvalues of $A_\xi^t A_\xi$, and we find corresponding frame of eigenvectors:

$$\sigma_0^2 = \sigma_1^2 = 0,\ u_0 = (-a_1, -a_2, a_3, 0),\ u_1 = (2a_1, 2a_2, 0, a_3)$$
$$\sigma_2^2 = \frac{1}{4}a_1^2 + \frac{1}{4}a_2^2 + \frac{1}{4}a_3^2,\ u_2 = (-a_2, a_1, 0, 0)$$
$$\sigma_3^2 = \frac{5}{4}a_1^2 + \frac{5}{4}a_2^2 + \frac{1}{4}a_3^2,\ u_3 = (a_1a_3, a_2a_3, a_1^2 + a_2^2, -2(a_1^2 + a_2^2))$$

Then to simplify some further calculations we find in the plain $(u_0, u_1)$ a geodesic vector $e_0$, wich also belongs to eigenvectors of $AA^T$ with eigenvalue zero, but $\nabla_{e_0} e_0 = 0$.

From the Koszul formula we can calculate all covariant derivatives on G.

Using Gramm-Schmidtt ortogonalization, we obtain new ortonormal basis, where

$$e_0 = (0, 0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$$
$$e_1 = \frac{(a_1, a_2, -\frac{1}{5}a_3, \frac{2}{5}a_3)}{\sqrt{a_1^2 + a_2^2 + \frac{1}{5}a_3^2}}$$
$$e_2 = \frac{(-a_2, a_1, 0, 0)}{\sqrt{a_1^2 + a_2^2}}$$
$$e_3 = \frac{(a_1a_3, a_2a_3, a_1^2 + a_2^2, -2(a_1^2 + a_2^2))}{\sqrt{(a_1^2 + a_2^2)(a_3^2 + 5(a_1^2 + a_2^2))}}$$

And we also know that

$$(\nabla_X A_Z)Y = \nabla_X(A_Z Y) - A_Z(\nabla_X Y)$$

or

$$(\nabla_X A_Z)Y = -A_{A_Z Y}X + A_Z(A_Y X)$$

or

$$(\nabla_X A_Z)Y = -\nabla_X(\nabla_Y Z) + \nabla_{\nabla_X Y} Z$$

And we use $R(X,Y)Z = (\nabla_X A_Z)Y - (\nabla_Y A_Z)X$ Riemann tensor.

# 4 Minimal vector fields in 4-dimentional case

If $\xi$ is globally defined, then $\xi$ is called minimal if $\xi(M)$ has minimal volume under variation of the vector field. Minimality of $\xi$ is equivalent to minimality of $\xi(M) \subset (T_1, g_S)$ as a submanifold. It follows that minimality of $\xi(M)$ make sense even if $\xi$ is defined locally.

In [3] it is stated that vector field $\xi$ is minimal when the mean curvature vector is zero, and is given such expression for mean curvature components:

$$H_k = \frac{1}{n\sqrt{1+\sigma_k^2}} \sum_{i=0}^{n-1} \frac{g((\nabla_{e_i} A_\xi)e_i, f_k) + g(A_\xi R(\xi, A_\xi e_i)e_i, f_k)}{1+\sigma_i^2}$$

$$H_k = \frac{1}{n\sqrt{1+\sigma_k^2}} \sum_{i=0}^{n-1} \frac{1}{1+\sigma_i^2} g\Big((\nabla_{e_i} A_\xi)e_i + A_\xi R(\xi, A_\xi e_i)e_i, f_k\Big)$$

**Theorem 1** *Let $G$ be a Riemannian manifold, equipped with Sasaki metric on it's tangent bundle, and let $\xi = (a_1, a_2, ..a_n)$ arbitrary unit vector field. Then $\xi$ is minimal if and only if:*

$$\sum_{i=0}^{n-1} \frac{1}{1+\sigma_i^2}\Big((\nabla_{e_i} A_\xi)e_i + A_\xi R(\xi, A_\xi e_i)e_i\Big) = \Big(\sum_{i=0}^{n-1} \frac{\sigma_i^2}{1+\sigma_i^2}\Big)\xi$$

*Where $\sigma_i$ are singular values for $\xi$, i.e. square roots of eigenvalues of $A_\xi^t A_\xi$, $e_i$ diagonalising basis, or left singular basis, i.e. orthonormed basis of eigenvectors for the operator $A_\xi^t A_\xi$, and $A_\xi X = -\nabla_X \xi$ - covariant derivation operator.*

**Proof.** We have defined $(f_k)_{k=0..n-1}$ is right singular basis for $\xi$, $f_0 = \xi$, and so $f_0$ should be ortogonal to all other $f_k, k = 1..n-1$

It means this sum should be proportional to $\xi$.

$$\sum_{i=0}^{n-1} \frac{1}{1+\sigma_i^2}\Big((\nabla_{e_i} A_\xi)e_i + A_\xi R(\xi, A_\xi e_i)e_i\Big) = \lambda\xi$$

Let's multiply by $\xi$:

$$\sum_{i=0}^{n-1} \frac{1}{1+\sigma_i^2}\Big\langle\Big((\nabla_{e_i} A_\xi)e_i + A_\xi R(\xi, A_\xi e_i)e_i\Big), \xi\Big\rangle = \lambda$$

But in this sum

$$\langle A_\xi R(\xi, A_\xi e_i)e_i, \xi\rangle = 0 \ \forall i = 0, 1..n-1$$

because, as we know, $\langle A_\xi X, \xi\rangle = 0$. It happens because $A_\xi X = -\nabla_X \xi$, and $\xi$ is unit vector field, $\langle \nabla_X \xi, \xi\rangle = 0$. So there have left only

$$\sum_{i=0}^{n-1} \frac{\langle (\nabla_{e_i} A_\xi)e_i, \xi\rangle}{1+\sigma_i^2} = \lambda$$

Again we use $A_\xi e_i = -\nabla_{e_i}\xi$, so $\langle(\nabla_{e_i}A_\xi)e_i, \xi\rangle = \langle A_\xi e_i, A_\xi e_i\rangle = \sigma_i^2$

So we obtain

$$\lambda = \sum_{i=0}^{n-1} \frac{\sigma_i^2}{1+\sigma_i^2}$$

So we have:

$$\sum_{i=0}^{n-1} \frac{1}{1+\sigma_i^2}\Big((\nabla_{e_i}A_\xi)e_i + A_\xi R(\xi, A_\xi e_i)e_i\Big) = \Big(\sum_{i=0}^{n-1} \frac{\sigma_i^2}{1+\sigma_i^2}\Big)\xi$$

■

**Remark 1** *We haven't used anything special for oscillator group or dimention 4 - we only used scalar product and covariant differentiation, so it works for any Riemannian manifold.*

**Theorem 2** *Let $G$ be the oscillator group, spanned by $E_1, E_2, E_3, E_4$, equipped with Sasaki metric on it's tangent bundle, and let $\xi = a_1E_1 + a_2E_2 + a_3E_3 + a_4E_4$ arbitrary unit vector field. Then $\xi$ is minimal if and only if:*

*$a_4 = 1, a_1 = a_2 = a_3 = 0$, which means $\xi = E_4$, or*

*$a_3 = 1, a_1 = a_2 = a_4 = 0$, which means $\xi = E_3$, or*

*$a_2 = a_4 = 0$, which means $\xi = a_1E_1 + a_2E_2$, $a_1^2 + a_2^2 = 1$,*

**Proof.** Than for our case we calculate that:

In order to find some covariant derivatives we use operator $A_\xi$, as we know, $A_\xi X = -\nabla_X\xi$, and so for every vector $X$, for example, for derivation our basis vectors, $e_i$, we should obtain matrices $A_{e_i}$ in the same way as $A_\xi$ and multiply the vector in which direction we want to derivate by corrresponding matrix, i.e. $A_{e_i}X = -\nabla_X e_i$.

$$A_{e_0} = \begin{pmatrix} 0 & \frac{\sqrt{5}}{5} & 0 & 0 \\ -\frac{\sqrt{5}}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So we see that

$$\nabla_{e_0}e_0 = \begin{pmatrix} 0 & \frac{\sqrt{5}}{5} & 0 & 0 \\ -\frac{\sqrt{5}}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} = 0$$

In the same way we obtain

$$A_{e_1} = \frac{1}{\sqrt{25a_1^2 + 25a_2^2 + 5a_3^2}} \begin{pmatrix} 0 & -\frac{a_3}{2} & \frac{5a_2}{2} & -5a_2 \\ -\frac{a_3}{2} & 0 & -\frac{5a_1}{2} & 5a_1 \\ \frac{5a_2}{2} & -\frac{5a_1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

And so $\nabla_{e_1} e_1$ is obtained easily as

$$\nabla_{e_1} e_1 = A_{e_1} e_1 = \frac{3a_3}{5} \frac{(-a_2, a_1, 0, 0)}{a_1^2 + a_2^2 + \frac{1}{5}a_3^2}$$

And so on,

$$\nabla_{e_2} e_2 = 0$$
$$\nabla_{e_3} e_3 = \frac{3a_3}{5 + a_3^2(a_1^2 + a_2^2)}(-a_2, a_1, 0, 0)$$

We can note that both $\nabla_{e_1} e_1$, $\nabla_{e_3} e_3$ are collinear to $e_2$.
Let's calculate $(\nabla_{e_i} A_\xi)e_i, i = 0..3$

$$(\nabla_{e_0} A_\xi)e_0 = 0$$
$$(\nabla_{e_1} A_\xi)e_1 = \frac{3a_3}{10(a_1^2 + a_2^2 + \frac{1}{5}a_3^2)}(a_1 a_3, a_2 a_3, -(a_1^2 + a_2^2), 0)$$
$$(\nabla_{e_2} A_\xi)e_2 = \frac{1}{4}(a_1, a_2, a_3, 0)$$

$$(\nabla_{e_3} A_\xi)e_3 = \frac{1}{4} \frac{1}{-5(a_1^2 + a_2^2) + a_3^2}(a_1(-25(a_1^2+a_2^2)+a_3^3), a_2(-25(a_1^2+a_2^2)+a_3^3),$$
$$a_3(-11(a_1^2 + a_2^2) + a_3^2), 0)$$

Then we find all $A_\xi R(\xi, A_\xi e_i)e_i$:
Besides, it appear that

$$R(\xi, A_\xi e_0)e_0 = R(\xi, A_\xi e_1)e_1 = R(\xi, A_\xi e_2)e_2 = 0$$

and only

$$R(\xi, A_\xi e_3)e_3 = \frac{1}{4}a_3(-a_2, a_1, 0, 0)$$

.

So

$$A_\xi R(\xi, A_\xi e_0)e_0 = A_\xi R(\xi, A_\xi e_1)e_1 = A_\xi R(\xi, A_\xi e_2)e_2 = 0$$

and only

$$A_\xi R(\xi, A_\xi e_3)e_3 = -\frac{1}{8}a_3(a_1 a_3, a_2 a_3, a_1^2 + a_2^2, 0)$$

**Remark 2** *For our case we can note some simplifications, provided with special properties of $E_4$, which on 4-dimentional oscillator group (in fact, not only 4-dimentional, but now we are examining this case only) behaves in a way, different to other basis vectors, and nearly doesn't take part in our calculation.*

In fact, $A_\xi$ does not depend on $a_4$, and if we mark that our sum has no projection on $E_4$, we can conclude that

$$\Big(\sum_{i=0}^{3} \frac{\sigma_i^2}{1 + \sigma_i^2}\Big)a_4 = 0$$

So either $a_4$ is 0 or $\sigma_i = 0$ (they are non-negative). If $\sigma_i = 0, \xi = (0,0,0,1)$, a parallel vector field. If $a_4 = 0$, while $\xi$ is unit we can use $a_1^2+a_2^2+a_3^2 = 1$.Then $\sigma_3^2 = \frac{5}{4} - a_3^2, \sigma_2^2 = \frac{1}{4}$.

Therefore we obtain $\sum_{i=0}^{3} \frac{1}{1+\sigma_i^2}(\nabla_{e_i} A_\xi)e_i + A_\xi R(\xi, A_\xi e_i)e_i$, which should be collinear to $\xi$

$$\sum_{i=0}^{3} \frac{1}{1+\sigma_i^2}(\nabla_{e_i} A_\xi)e_i + A_\xi R(\xi, A_\xi e_i)e_i = \begin{pmatrix} \frac{a_1(a_3^2(a_4^2-1)+10a_4^2-34)}{(a_3^2+5(a_1^2+a_2^2+1))(a_4^2-5)} \\ \frac{a_2(a_3^2(a_4^2-1)+10a_4^2-34)}{(a_3^2+5(a_1^2+a_2^2+1))(a_4^2-5)} \\ \frac{a_3(a_4^2(a_4^2+a_3^2)-a_3^2-9)}{(a_3^2+5(a_1^2+a_2^2+1))(a_4^2-5)} \\ 0 \end{pmatrix}$$

Using our simplifications we obtain:

$$\sum_{i=0}^{3} \frac{1}{1+\sigma_i^2}(\nabla_{e_i} A_\xi)e_i + A_\xi R(\xi, A_\xi e_i)e_i = \begin{pmatrix} \frac{a_1(a_3^2+34)}{5(a_3^2+5(a_1^2+a_2^2+1))} \\ \frac{a_2(a_3^2+34)}{5(a_3^2+5(a_1^2+a_2^2+1))} \\ \frac{a_3(a_3^2+9)}{5(a_3^2+5(a_1^2+a_2^2+1))} \\ 0 \end{pmatrix}$$

while unit vector field $\xi = (a_1, a_2, a_3, a_4)$.

As we already know, we have such cases: either $\xi = (0,0,0,1) = E_4$ or $a_4 = 0$. In that case we have $\xi = (a_1, a_2, a_3, 0)$ is collinear to

$$\Big(\frac{a_1(a_3^2+34)}{5(a_3^2+5(2-a_3^2))}, \frac{a_2(a_3^2+34)}{5(a_3^2+5(2-a_3^2))}, \frac{a_3(a_3^2+9)}{5(a_3^2+5(2-a_3^2))}, 0\Big)$$

Here we see that $a_1 = a_2 = 0$, and $\xi = (0,0,1,0) = E_3$, or $a_3 = 0$ and $\xi = (a_1, a_2, 0, 0) \in (E_1 \hat{,} E_2)$, $a_1^2 + a_2^2 = 1$.

■

# 5 Totally geodesic vector field in 4-dimentional case

Vector field $\xi$ is called totally geodesic when the image of $\xi : M \to T_1M$ is totally geodesic submanifold in the unit tangent bundle. As it was said in [1], we are looking for $\xi$, such that the second fundamental form

$$\Omega_\xi(X,Y) = Hess_\xi(X,Y) + A_\xi Hm_\xi(X,Y) - g(A_\xi X, A_\xi Y) = 0$$

for any $X, Y \in \mathfrak{X}(M)$.

Here we know that

$$Hess_\xi(X,Y) = \frac{1}{2}((\nabla_X A_\xi)Y + (\nabla_Y A_\xi)X)$$

is rough hessian and

$$Hm_\xi(X,Y) = \frac{1}{2}(R(A_\xi X, \xi)Y + R(A_\xi Y, \xi)X)$$

is harmonisity tensor.

**Theorem 3** *Let $G$ be the oscillator group, spanned by $E_1, E_2, E_3, E_4$, equipped with Sasaki metric on it's tangent bundle, then the only totally geodesic unit vector field is $E_4$.*

**Proof.** While $\Omega_\xi(X, Y)$ must be 0 on all $X, Y$, let's calculate it at least on some basis vectors. For example, we obtain

$$\Omega_\xi(E_4, E_4) = -(a_1(a_1^2 + a_2^2 - 1), a_2(a_1^2 + a_2^2 - 1), a_3(a_1^2 + a_2^2), a_4(a_1^2 + a_2^2))$$

So we see that either $a_4 = a_3 = 0$ or $a_1^2 + a_2^2 = 0$.
Then considering $\Omega_\xi(E_2, E_4)$ and $\Omega_\xi(E_1, E_4)$, we see that

$$\Omega_\xi(E_2, E_4) = \frac{1}{16}(9a_1a_2a_3, a_3(2a_1^2 + 11a_2^2), -a_2(3(a_1^2 + a_2^2) - 8a_3^2 + 8), 8a_2a_3a_4)$$

and

$$\Omega_\xi(E_1, E_4) = \frac{1}{16}(a_3(11a_1^2 + 2a_2^2), 9a_1a_2a_3, -a_1(3(a_1^2 + a_2^2) - 8a_3^2 + 8), 8a_1a_3a_4)$$

So if $a_4 = a_3 = 0$ then also $a_2 = 0$ and $a_1 = 0$, no solution.
So there should be always $a_1^2 + a_2^2 = 0$.
From $\Omega_\xi(E_4, E_4)$ we see that it follows then also $a_1 = 0$ and $a_2 = 0$.
We calculate also $\Omega_\xi(E_1, E_3)$:

$$\Omega_\xi(E_1, E_3) = -\frac{1}{16}(2a_3(11a_1^2 + a_2^2 - a_3^2 + 4), 5a_1a_2a_3, -\frac{3}{2}a_1(a_1^2 + a_2^2 - 3a_3 + 4), 4a_1a_3a_4)$$

from which we see that also $a_3 = 0$, and so we have to check only if $\xi = E_4 = (0, 0, 0, 1)$ gives us solution. Simply checking all $\Omega_{E_4}(E_i, E_j), i, j = 1..4$ we see they are all zero. But while all components of $\Omega$ are linear, it's enough to check it on basis.

■

# 6 Minimal and geodesic vector fields in case $G_{2n+2}(1)$

Consider $2n + 2$-dimentional oscillator group $G = G(1)$, (i.e., we firstly consider simplier case, when $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 1$) which is equipped with Sasaki metric on it's tangent bundle and in standart basis $X_1, \cdots, X_n, Y_1, \cdots, Y_n, P, Q$ it has only following non-vanishing Lie brackets:

$$[X_i, Y_j] = \delta_{ij}P,\ [Q, X_j] = Y_j,\ [Q, Y_j] = -X_j,\ 1 \leq i, j \leq n$$

Define $\xi$ arbitrary unit vector field $\xi = (a_1, \cdots, a_{2n+2})$, operator $A_\xi X = -\nabla_X \xi$. By Koszul formula we can calculate all covariant derivatives.

Applying it to the standart basis, we obtain matrix of this linear operator:

$$A_\xi = \begin{pmatrix} 0 & \cdots & 0 & \frac{a_{2n+1}}{2} & \cdots & 0 & \frac{a_{n+1}}{2} & -a_{n+1} \\ \vdots & \ddots & \vdots & 0 & \ddots & 0 & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \frac{a_{2n+1}}{2} & \frac{a_{2n}}{2} & -a_{2n} \\ -\frac{a_{2n+1}}{2} & \cdots & 0 & 0 & \cdots & 0 & -\frac{a_1}{2} & a_1 \\ 0 & \ddots & 0 & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & -\frac{a_{2n+1}}{2} & 0 & \cdots & 0 & -\frac{a_n}{2} & a_n \\ \frac{a_{n+1}}{2} & \cdots & \frac{a_{2n}}{2} & -\frac{a_1}{2} & \cdots & -\frac{a_n}{2} & 0 & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

We call $\xi_X = \sum_{i=1}^n a_i X_i$ and $\xi_Y = \sum_{i=1}^n a_{n+i} Y_i$, and $I_n$ the identity matrix $n \times n$.

Now we can write

$$A_\xi = \begin{pmatrix} 0 & \frac{a_{2n+1}}{2} I_n & \frac{1}{2}\xi_Y & -\xi_Y \\ -\frac{a_{2n+1}}{2} I_n & 0 & -\frac{1}{2}\xi_X & \xi_X \\ \frac{1}{2}\xi_Y^t & -\frac{1}{2}\xi_X^t & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Our group G is direct product of 1-dimentional subgroup, associated with the last basis vector, Q, and $2n+1$-dimentional subgroup, spanned by

$$X_1, \cdots, X_n, Y_1, \cdots, Y_n, P$$

and taking into attention the results of 4-dimentional case in previous chapters and $2n+2$ dimentional case, considered in [6], we will look for the answer $\xi \subset Q^\perp$. To simplify some further calculations, let us look at geodesic vectors in subspace $Q^\perp$. As we know, geodesic vector field has $\nabla_\xi \xi = 0$, and, while we assume $a_{2n+2} = 0$, we are looking for $\xi$ s.t. $A_\xi \xi = 0$.

$$\nabla_\xi \xi = -\begin{pmatrix} 0 & \frac{a_{2n+1}}{2} I_n & \frac{1}{2}\xi_Y & -\xi_Y \\ -\frac{a_{2n+1}}{2} I_n & 0 & -\frac{1}{2}\xi_X & \xi_X \\ \frac{1}{2}\xi_Y^t & -\frac{1}{2}\xi_X^t & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_X \\ \xi_Y \\ a_{2n+1} \\ a_{2n+2} \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} (a_{2n+1} - a_{2n+2})\xi_Y \\ -(a_{2n+1} - a_{2n+2})\xi_X \\ 0 \\ 0 \end{pmatrix}$$

Let us define a special vector

$$\phi(\xi) = \phi(\sum_{i=1}^n (a_i X_i + a_{n+i} Y_i) + a_{2n+1} P + a_{2n+2} Q) = \sum_{i=1}^n (a_i Y_i - a_{n+i} X_i)$$

We can simplify our notation:

$$\nabla_\xi \xi = -\frac{1}{2}(a_{2n+1} - a_{2n+2})\phi(\xi) = 0$$

While we decided to look for our solution in $\xi \in Q^{\perp}$ it mean we need either $\xi = P$, $a_i = 0, i = 1..2n, 2n+2$ or $a_{2n+1} = a_{2n+2} = 0$, $\xi \in P^{\perp} \subset Q^{\perp}$(looks similar to 4-dimentional case (n=1), when we had either $E_3$, or $E_4$, or any unit vector in the plane of $E_1, E_2$). So now we can assume $a_{2n+1} = a_{2n+2} = 0$, and rewrite

$$A_\xi = \begin{pmatrix} 0 & 0 & \frac{1}{2}\xi_Y & -\xi_Y \\ 0 & 0 & -\frac{1}{2}\xi_X & \xi_X \\ \frac{1}{2}\xi_Y^t & -\frac{1}{2}\xi_X^t & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2}\phi(\xi) & \phi(\xi) \\ -\frac{1}{2}\phi(\xi)^t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Now we know all vectors $\xi \in P^{\perp} \subset Q^{\perp}$ are geodesic. Let us check their minimality applying Theorem 1.

In order to find singular values and singular basis for $\xi$ we calculate matrix $A_\xi^t A_\xi$ and find it's eigenvalues and eigenvectors:

$$A_\xi^t A_\xi = \begin{pmatrix} \frac{1}{4}\xi_Y \otimes \xi_Y & -\frac{1}{4}\xi_Y \otimes \xi_X^t & 0 & 0 \\ -\frac{1}{4}\xi_X \otimes \xi_Y^t & \frac{1}{4}\xi_X \otimes \xi_X^t & 0 & 0 \\ 0 & 0 & \frac{1}{4} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{4}\phi(\xi)\phi(\xi)^t & 0 & 0 \\ 0 & \frac{1}{4} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}$$

Eigenvalues and eigenvectors of $A_\xi^t A_\xi$ are:

$$\sigma_i^2 = 0,\ i = 1..2n-1,\ \sigma_{2n}^2 = \frac{1}{4}$$
$$\sigma_{2n+1}^2 = \frac{5}{4},\ \sigma_{2n+2}^2 = 0$$
$$e_i = \text{ortonormal basis} \subset \phi(\xi)^{\perp} \subset P^{\perp} \subset Q^{\perp} \text{ for } i = 1..2n-1$$
$$e_{2n} = \phi(\xi) = \sum_{i=1}^{n} (a_i Y_i - a_{n+i} X_i)$$
$$e_{2n+1} = \frac{1}{\sqrt{5}}(-P + 2Q),\ e_{2n+2} = \frac{1}{\sqrt{5}}(2P + Q)$$

For the minimality of vector field $\xi$ we need

$$\sum_{i=0}^{2n+1} \frac{1}{1+\sigma_i^2}\Big((\nabla_{e_i} A_\xi)e_i + A_\xi R(\xi, A_\xi e_i)e_i\Big) = \Big(\sum_{i=0}^{2n+1} \frac{\sigma_i^2}{1+\sigma_i^2}\Big)\xi$$

As we can easily see from the structure of $A_\xi$ matrix, all covariant derivatives in $P^{\perp} \subset Q^{\perp}$ are zero, i.e.

$$\forall X \in P^{\perp} \subset Q^{\perp} \quad \nabla_X X = 0$$

And so we can note that

$$A_{e_i} e_i = -\nabla_{e_i} e_i = 0$$
$$(\nabla_{e_i} A_\xi)e_i = \nabla_{e_i}(A_\xi e_i) - A_\xi(\nabla_{e_i} e_i) = 0$$
$$\forall i = 1..2n-1$$

It means we need to check our sum only on the last three vector of singular basis, all other components are zero. So we calculate

$$\nabla_{e_{2n}} e_{2n} = -A_{e_{2n}} e_{2n} = -\begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & -\frac{a_1}{2} & a_1 \\ \vdots & \ddots & \vdots & 0 & \ddots & 0 & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{-a_n}{2} & a_n \\ 0 & \cdots & 0 & 0 & \cdots & 0 & -\frac{a_{n+1}}{2} & a_{n+1} \\ 0 & \ddots & 0 & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & -\frac{a_{2n}}{2} & a_{2n} \\ -\frac{a_1}{2} & \cdots & -\frac{a_n}{2} & -\frac{a_{n+1}}{2} & \cdots & -\frac{a_{2n}}{2} & 0 & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -a_{n+1} \\ \vdots \\ -a_{2n} \\ a_1 \\ \vdots \\ a_n \\ 0 \\ 0 \end{pmatrix} = 0$$

$$A_\xi e_{2n} = \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{a_{n+1}}{2} & -a_{n+1} \\ \vdots & \ddots & \vdots & 0 & \ddots & 0 & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{a_{2n}}{2} & -a_{2n} \\ 0 & \cdots & 0 & 0 & \cdots & 0 & -\frac{a_1}{2} & a_1 \\ 0 & \ddots & 0 & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & -\frac{a_n}{2} & a_n \\ \frac{a_{n+1}}{2} & \cdots & \frac{a_{2n}}{2} & -\frac{a_1}{2} & \cdots & -\frac{a_n}{2} & 0 & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -a_{n+1} \\ \vdots \\ -a_{2n} \\ a_1 \\ \vdots \\ a_n \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -\frac{1}{2} \\ 0 \end{pmatrix}$$

$$(\nabla_{e_{2n}} A_\xi) e2n = \nabla_{e_{2n}}(A_\xi e_{2n}) - A_\xi(\nabla_{e_{2n}} e_{2n}) =$$

$$= \begin{pmatrix} 0 & -\frac{1}{4}I_n & 0 & 0 \\ \frac{1}{4}I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -a_{n+1} \\ \vdots \\ -a_{2n} \\ a_1 \\ \vdots \\ a_n \\ 0 \\ 0 \end{pmatrix} = \frac{1}{4}\xi$$

$$A_\xi e_{2n+1} = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{a_{n+1}}{2} & -a_{n+1} \\ \vdots & \ddots & \vdots & 0 & \ddots & 0 & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{a_{2n}}{2} & -a_{2n} \\ 0 & \cdots & 0 & 0 & \cdots & 0 & -\frac{a_1}{2} & a_1 \\ 0 & \ddots & 0 & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & -\frac{a_n}{2} & a_n \\ \frac{a_{n+1}}{2} & \cdots & \frac{a_{2n}}{2} & -\frac{a_1}{2} & \cdots & -\frac{a_n}{2} & 0 & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 2 \end{pmatrix} =$$

$$= -\nabla_{e_{2n+1}}\xi = -\frac{5}{2\sqrt{5}}\phi(\xi)$$

$$\nabla_{e_{2n+1}} e_{2n+1} = -\frac{1}{\sqrt{5}} \begin{pmatrix} 0 & -\frac{1}{2}I_n & 0 & 0 \\ \frac{1}{2}I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 2 \end{pmatrix} = 0$$

$$(\nabla_{e_{2n+1}} A_\xi) e_{2n+1} = \nabla_{e_{2n+1}}(A_\xi e_{2n+1}) - A_\xi(\nabla_{e_{2n+1}} e_{2n+1}) = -\frac{5}{2\sqrt{5}} \nabla_{e_{2n+1}}\phi(\xi) =$$

$$= \frac{\sqrt{5}}{2} A_{\phi(\xi)} e_{2n+1} = \frac{1}{2} \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & -\frac{a_1}{2} & a_1 \\ \vdots & \ddots & \vdots & 0 & \ddots & 0 & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{-a_n}{2} & a_n \\ 0 & \cdots & 0 & 0 & \cdots & 0 & -\frac{a_{n+1}}{2} & a_{n+1} \\ 0 & \ddots & 0 & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & -\frac{a_{2n}}{2} & a_{2n} \\ -\frac{a_1}{2} & \cdots & -\frac{a_n}{2} & -\frac{a_{n+1}}{2} & \cdots & -\frac{a_{2n}}{2} & 0 & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 2 \end{pmatrix} = \frac{5}{4}\xi$$

$$A_\xi e_{2n+2} = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{a_{n+1}}{2} & -a_{n+1} \\ \vdots & \ddots & \vdots & 0 & \ddots & 0 & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{a_{2n}}{2} & -a_{2n} \\ 0 & \cdots & 0 & 0 & \cdots & 0 & -\frac{a_1}{2} & a_1 \\ 0 & \ddots & 0 & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & -\frac{a_n}{2} & a_n \\ \frac{a_{n+1}}{2} & \cdots & \frac{a_{2n}}{2} & -\frac{a_1}{2} & \cdots & -\frac{a_n}{2} & 0 & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 2 \\ 1 \end{pmatrix} = 0$$

$$\nabla_{e_{2n+2}} e_{2n+2} = 0$$

$$(\nabla_{e_{2n+2}} A_\xi) e_{2n+2} = \nabla_{e_{2n+2}}(A_\xi e_{2n+2}) - A_\xi(\nabla_{e_{2n+2}} e_{2n+2}) = 0$$

First term is 0 for all $i$ exept $i = 2n$ and $i = 2n+1$, where it is parallel to $\xi$, and we have to calculate Riemannian tensor now.

$$R(\xi, A_\xi e_{2n+1})e_{2n+1} = -A_{A_{e_{2n+1}} A_\xi e_{2n+1}}\xi + A_{e_{2n+1}}(A_{A_\xi e_{2n+1}}\xi) + A_{A_{e_{2n+1}}\xi} A_\xi e_{2n+1} -$$
$$- A_{e_{2n+1}}(A_\xi A_\xi e_{2n+1}) = \frac{3}{2\sqrt{5}} A_{A_{e_{2n+1}}\phi(\xi)}\xi - \frac{3}{2\sqrt{5}} A_{e_{2n+1}}(A_{\phi(\xi)}\xi) - \frac{3}{2\sqrt{5}} A_{A_{e_{2n+1}}\xi}\phi(\xi) +$$
$$+ \frac{3}{2\sqrt{5}} A_{e_{2n+1}}(A_\xi \phi(\xi)) = 0$$
$$R(\xi, A_\xi e_{2n+2})e_{2n+2} = 0$$

For our sum there have left only two terms, and it is parallel to $\xi$ for all $a_i$, and it is

$$\sum_{i=0}^{2n+1} \frac{1}{1+\sigma_i^2}\Big((\nabla_{e_i} A_\xi)e_i + A_\xi R(\xi, A_\xi e_i)e_i\Big) = \frac{\frac{1}{4}\xi}{1+\frac{1}{4}} + \frac{\frac{5}{4}\xi}{1+\frac{5}{4}} =$$
$$= \Big(\frac{\frac{1}{4}}{1+\frac{1}{4}} + \frac{\frac{5}{4}}{1+\frac{5}{4}}\Big)\xi = \Big(\sum_{i=0}^{2n+1} \frac{\sigma_i^2}{1+\sigma_i^2}\Big)\xi$$

so all $\sum_{i=1}^{n}(a_i X_i + a_{n+i} Y_i)$, $\sum_{i=1}^{2n} a_i^2 = 1$ are minimal.

# 7 Minimal and geodesic vector fields in case $G_{2n+2}(\lambda)$

Now let us consider broader case, when $2n+2$-dimentional oscillator group $G = G_{2n+2}(\lambda)$, (i.e., we consider the case, when $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \lambda \neq 1$) again equipped with Sasaki metric on it's tangent bundle and in standart basis $X_1, \cdots, X_n, Y_1, \cdots, Y_n, P, Q$ it has only following non-vanishing Lie brackets:

$$[X_i, Y_j] = \delta_{ij} P,\ [Q, X_j] = \lambda_j Y_j,\ [Q, Y_j] = -\lambda_j X_j,\ 1 \le i, j \le n$$

Define $\xi$ arbitrary unit vector field $\xi = (a_1, \cdots, a_{2n+2})$, operator $A_\xi X = -\nabla_X \xi$. By Koszul formula we can calculate all covariant derivatives.

Applying it to the standart basis, we obtain matrix of this linear operator:

$$A_\xi = \begin{pmatrix} 0 & \cdots & 0 & \frac{a_{2n+1}}{2} & \cdots & 0 & \frac{a_{n+1}}{2} & -\lambda a_{n+1} \\ \vdots & \ddots & \vdots & 0 & \ddots & 0 & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \frac{a_{2n+1}}{2} & \frac{a_{2n}}{2} & -\lambda a_{2n} \\ -\frac{a_{2n+1}}{2} & \cdots & 0 & 0 & \cdots & 0 & -\frac{a_1}{2} & \lambda a_1 \\ 0 & \ddots & 0 & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & -\frac{a_{2n+1}}{2} & 0 & \cdots & 0 & -\frac{a_n}{2} & \lambda a_n \\ \frac{a_{n+1}}{2} & \cdots & \frac{a_{2n}}{2} & -\frac{a_1}{2} & \cdots & -\frac{a_n}{2} & 0 & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

**Theorem 4** *Let $G$ be $2n+2$-dimentional oscillator group, spanned by*

$$X_1, \cdots, X_n, Y_1, \cdots, Y_n, P, Q$$

*equipped with Sasaki metric on it's tangent bundle, and let*

$$\xi = \sum_{i=1}^{n} (a_i X_i + a_{n+i} Y_i) + a_{2n+1} P + a_{2n+2} Q$$

*arbitrary unit vector field.*

*Then if $\xi$ is either $P$ or $Q$ or$\sum_{i=1}^n (a_i X_i + a_{n+i} Y_i)$, $\sum_{i=1}^{2n} a_i^2 = 1$ it is gedesic and minimal. There are no other geodesic unit vector fields on $G$.*

**.** We are going to use absolutely same procedure and same notations. Only a little have changed, indeed. For the same reasons, we are looking firstly for $\xi \in Q^\perp$, $\nabla_\xi \xi = 0$:

$$\nabla_\xi \xi = - \begin{pmatrix} 0 & \frac{a_{2n+1}}{2} I_n & \frac{1}{2}\xi_Y & -\lambda \xi_Y \\ -\frac{a_{2n+1}}{2} I_n & 0 & -\frac{1}{2}\xi_X & \lambda \xi_X \\ \frac{1}{2}\xi_Y^t & -\frac{1}{2}\xi_X^t & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_X \\ \xi_Y \\ a_{2n+1} \\ a_{2n+2} \end{pmatrix} =$$

$$= -\frac{1}{2} \begin{pmatrix} (a_{2n+1} - \lambda a_{2n+2})\xi_Y \\ -(a_{2n+1} - \lambda a_{2n+2})\xi_X \\ 0 \\ 0 \end{pmatrix}$$

So we have $\xi \in P^\perp \subset Q^\perp$ or $P$, or $Q$ as all geodesic again. Let's apply Theorem 1. We need singular values and singular basis.

In order to find singular values and singular basis for $\xi$, we calculate matrix $A_\xi^t A_\xi$ and find it's eigenvalues and eigenvectors:

$$A_\xi^t A_\xi = \begin{pmatrix} \frac{1}{4}\xi_Y \otimes \xi_Y & -\frac{1}{4}\xi_Y \otimes \xi_X^t & 0 & 0 \\ -\frac{1}{4}\xi_X \otimes \xi_Y^t & \frac{1}{4}\xi_X \otimes \xi_X^t & 0 & 0 \\ 0 & 0 & \frac{1}{4} & -\frac{1}{2}\lambda \\ 0 & 0 & -\frac{1}{2}\lambda & \lambda^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4}\phi(\xi)\phi(\xi)^t & 0 & 0 \\ 0 & \frac{1}{4} & -\frac{1}{2}\lambda \\ 0 & -\frac{1}{2}\lambda & \lambda^2 \end{pmatrix}$$

Eigenvalues and eigenvectors of $A_\xi^t A_\xi$ are:

$$\sigma_i^2 = 0,\, i = 1..2n-1,\, \sigma_{2n}^2 = \frac{1}{4}$$
$$\sigma_{2n+1}^2 = \lambda^2 + \frac{1}{4},\, \sigma_{2n+2}^2 = 0$$
$$e_i = \text{ortonormal basis} \subset \phi(\xi)^\perp \subset P^\perp \subset Q^\perp \text{ for } i = 1..2n-1$$
$$e_{2n} = \phi(\xi) = \sum_{i=1}^{n} (a_i Y_i - a_{n+i} X_i)$$
$$e_{2n+1} = \frac{1}{\sqrt{4\lambda^2+1}}(-P + 2\lambda Q),\, e_{2n+2} = \frac{1}{\sqrt{4\lambda^2+1}}(2\lambda P + Q)$$

Only the last two vectors have changed and only one singular value.
For the minimality of vector field $\xi$ we need

$$\sum_{i=0}^{2n+1} \frac{1}{1+\sigma_i^2}\Big((\nabla_{e_i} A_\xi)e_i + A_\xi R(\xi, A_\xi e_i)e_i\Big) = \Big(\sum_{i=0}^{2n+1} \frac{\sigma_i^2}{1+\sigma_i^2}\Big)\xi$$

Again first 2n-1 terms are zero, let's calculate the last three ones:

$$\nabla_{e_{2n}} e_{2n} = -A_{e_{2n}} e_{2n} =$$
$$= -\begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & -\frac{a_1}{2} & \lambda a_1 \\ \vdots & \ddots & \vdots & 0 & \ddots & 0 & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{-a_n}{2} & \lambda a_n \\ 0 & \cdots & 0 & 0 & \cdots & 0 & -\frac{a_{n+1}}{2} & \lambda a_{n+1} \\ 0 & \ddots & 0 & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & -\frac{a_{2n}}{2} & \lambda a_{2n} \\ -\frac{a_1}{2} & \cdots & -\frac{a_n}{2} & -\frac{a_{n+1}}{2} & \cdots & -\frac{a_{2n}}{2} & 0 & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -a_{n+1} \\ \vdots \\ -a_{2n} \\ a_1 \\ \vdots \\ a_n \\ 0 \\ 0 \end{pmatrix} = 0$$

$$A_\xi e_{2n} = \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{a_{n+1}}{2} & -\lambda a_{n+1} \\ \vdots & \ddots & \vdots & 0 & \ddots & 0 & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{a_{2n}}{2} & -\lambda a_{2n} \\ 0 & \cdots & 0 & 0 & \cdots & 0 & -\frac{a_1}{2} & \lambda a_1 \\ 0 & \ddots & 0 & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & -\frac{a_n}{2} & \lambda a_n \\ \frac{a_{n+1}}{2} & \cdots & \frac{a_{2n}}{2} & -\frac{a_1}{2} & \cdots & -\frac{a_n}{2} & 0 & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -a_{n+1} \\ \vdots \\ -a_{2n} \\ a_1 \\ \vdots \\ a_n \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -\frac{1}{2} \\ 0 \end{pmatrix}$$

$$(\nabla_{e_{2n}} A_\xi) e_2 n = \nabla_{e_{2n}}(A_\xi e_{2n}) - A_\xi(\nabla_{e_{2n}} e_{2n}) =$$

$$= \begin{pmatrix} 0 & -\frac{1}{4}I_n & 0 & 0 \\ \frac{1}{4}I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -a_{n+1} \\ \vdots \\ -a_{2n} \\ a_1 \\ \vdots \\ a_n \\ 0 \\ 0 \end{pmatrix} = \frac{1}{4}\xi$$

For this term no changes, as expected. Let's check last two ones:

$$A_\xi e_{2n+1} = \frac{1}{\sqrt{4\lambda^2+1}} \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{a_{n+1}}{2} & -\lambda a_{n+1} \\ \vdots & \ddots & \vdots & 0 & \ddots & 0 & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{a_{2n}}{2} & -\lambda a_{2n} \\ 0 & \cdots & 0 & 0 & \cdots & 0 & -\frac{a_1}{2} & \lambda a_1 \\ 0 & \ddots & 0 & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & -\frac{a_n}{2} & \lambda a_n \\ \frac{a_{n+1}}{2} & \cdots & \frac{a_{2n}}{2} & -\frac{a_1}{2} & \cdots & -\frac{a_n}{2} & 0 & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 2\lambda \end{pmatrix} =$$

$$= -\nabla_{e_{2n+1}}\xi = -\frac{\frac{1}{2}+2\lambda^2}{\sqrt{4\lambda^2+1}}\phi(\xi)$$

$$\nabla_{e_{2n+1}} e_{2n+1} = -\frac{1}{\sqrt{4\lambda^2+1}} \begin{pmatrix} 0 & -\frac{1}{2}I_n & 0 & 0 \\ \frac{1}{2}I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 2\lambda \end{pmatrix} = 0$$

$$(\nabla_{e_{2n+1}}A_\xi)e_{2n+1} = \nabla_{e_{2n+1}}(A_\xi e_{2n+1}) - A_\xi(\nabla_{e_{2n+1}}e_{2n+1}) =$$
$$= -\frac{\frac{1}{2}+2\lambda^2}{\sqrt{4\lambda^2+1}}\nabla_{e_{2n+1}}\phi(\xi) == \frac{\sqrt{4\lambda^2+1}}{2}A_{\phi(\xi)}e_{2n+1} =$$
$$= \frac{1}{2}\begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & -\frac{a_1}{2} & \lambda a_1 \\ \vdots & \ddots & \vdots & 0 & \ddots & 0 & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{-a_n}{2} & \lambda a_n \\ 0 & \cdots & 0 & 0 & \cdots & 0 & -\frac{a_{n+1}}{2} & \lambda a_{n+1} \\ 0 & \ddots & 0 & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & -\frac{a_{2n}}{2} & \lambda a_{2n} \\ -\frac{a_1}{2} & \cdots & -\frac{a_n}{2} & -\frac{a_{n+1}}{2} & \cdots & -\frac{a_{2n}}{2} & 0 & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 2\lambda \end{pmatrix} = \frac{(4\lambda^2+1)}{4}\xi$$

$$A_\xi e_{2n+2} = \frac{1}{\sqrt{4\lambda^2+1}}\begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{a_{n+1}}{2} & -\lambda a_{n+1} \\ \vdots & \ddots & \vdots & 0 & \ddots & 0 & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{a_{2n}}{2} & -\lambda a_{2n} \\ 0 & \cdots & 0 & 0 & \cdots & 0 & -\frac{a_1}{2} & \lambda a_1 \\ 0 & \ddots & 0 & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & -\frac{a_n}{2} & \lambda a_n \\ \frac{a_{n+1}}{2} & \cdots & \frac{a_{2n}}{2} & -\frac{a_1}{2} & \cdots & -\frac{a_n}{2} & 0 & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 2\lambda \\ 1 \end{pmatrix} = 0$$
$$\nabla_{e_{2n+2}}e_{2n+2} = 0$$
$$(\nabla_{e_{2n+2}}A_\xi)e_{2n+2} = \nabla_{e_{2n+2}}(A_\xi e_{2n+2}) - A_\xi(\nabla_{e_{2n+2}}e_{2n+2}) = 0$$

All terms with Riemannian tensor are again zeros.

For our sum there have left only two terms, and it is parallel to $\xi$ for all $a_i$, and it is

$$\sum_{i=0}^{2n+1}\frac{1}{1+\sigma_i^2}\Big((\nabla_{e_i}A_\xi)e_i + A_\xi R(\xi, A_\xi e_i)e_i\Big) = \frac{\frac{1}{4}\xi}{1+\frac{1}{4}} + \frac{\frac{(4\lambda^2+1)}{4}\xi}{\lambda^2+\frac{5}{4}} =$$
$$= \Big(\frac{\frac{1}{4}}{1+\frac{1}{4}} + \frac{\lambda^2+\frac{1}{4}}{\lambda^2+\frac{5}{4}}\Big)\xi = \Big(\sum_{i=0}^{2n+1}\frac{\sigma_i^2}{1+\sigma_i^2}\Big)\xi$$

so all $\sum_{i=1}^{n}(a_iX_i + a_{n+i}Y_i)$, $\sum_{i=1}^{2n}a_i^2 = 1$ are minimal, and the result does not depend on $\lambda$.

■

# 8 Results

In this thesis we have found all minimal and all totally geodesic unit left-invariant vector fields in the case of 4-dimentional oscillator group. We have also found all geodesic vector fields, and the set of all minimal unit left-invariant vector fields, geodesic and orthogonal to the one-dimensional subspace spanned by the last, special basis vector in the multidimensional case and we also formulate a theorem for checking the minimality of a unit vector field on the Sasaki metric on the tangent bundle of a Riemannian manifold.

# 9 References

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