ZERO SETS OF ENTIRE ABSOLUTELY MONOTONIC FUNCTIONS

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Introduction

This paper is devoted to the problem of characterization of the class of subsets of C which can serve as zero sets of entire absolutely monotonic functions. This problem was posed in [1] and has been solved for finite sets there. Here we give the

By the definition, an entire absolutely monotonic function f is an entire function representable in the form

$$f(z) = \int_0^\infty e^{zu} P(du), \tag{0.1}$$

where P is a nonnegative finite Borel measure on R^+ and the integral converges absolutely for each $z \in C$. By the well-known S. Bernstein's theorem [2], the class of such functions can be defined as the class of entire functions f such that

$$f^{(k)}(x) > 0, \quad \forall k \in \mathbf{N} \bigcup \{0\},\ \forall x \in R. \tag{0.2}$$

Entire absolutely monotonic functions form a proper subclass of the class of entire functions representable in the form

$$f(z) = \int_{-\infty}^\infty e^{zu} P(du), \tag{0.3}$$

where P is a finite non-negative Borel measure on R and the integral converges absolutely on C. The zero sets corresponding to the class described by (0.3) were completely characterized in [3]. This characterization is the following:

Theorem A *([3]). A set $E \subset C$ without finite accumulation points is the zero set of a function of the form (0.3) iff the following conditions are satisfied:*

(a)

$$E \bigcap R = \emptyset, \quad b \in E \Leftrightarrow \bar{b} \in E \tag{0.4}$$

(multiplicities of b and $\bar{b}$ are equal);

(b) for every $H > 0$

$$\log n(r, H) = o(r), \quad r \to \infty, \tag{0.5}$$

holds, where

$$n(r, H) := \#\{z : z \in E, |Imz| \le r, |Rez| \le H\} \tag{0.6}$$

(points of E are counted with their multiplicities).

Sure, zero sets of entire absolutely monotonic functions form a subclass of sets described in the above theorem. On the other hand, it is evident that entire absolutely monotonic functions form a subclass of the class of entire functions bounded in each half-plane of the kind

$$C_\omega := \{z : \mathrm{Re} z \le \omega\}, \quad \omega \in R. \tag{0.7}$$

Therefore the characterization of the zero sets of entire functions bounded in each half-plane C_ω is of interest. We give a complete characterization of these sets:

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Theorem 1 *A set $E = \{b_k\}_{k=1}^{\infty} \subset C$ without finite accumulation points is the zero set of an entire function bounded in C_ω, $\forall \omega \in R$, iff*

$$\sum_{b_k \in E \bigcap C_\omega} \frac{|Reb_k| + 1}{|b_k|^2 + 1} < \infty, \quad \forall \omega \in R. \tag{0.8}$$

Note that the necessity of the condition (0.8) is an easy consequence of the well-known Blaschke condition for a half-plane. It can be easily shown that the condition (0.8) implies (b) in Theorem A.

It turned out that, if we add (0.8) to the conditions of Theorem A, we do not obtain the complete charactreization of zero sets of entire absolutely monotonic functions. In [1] it was mentioned the following necessary condition not depending on all previous ones:

$$\text{dist}(x, E) \to +\infty, \quad x \to -\infty. \tag{0.9}$$

I.V. Ostrovskii showed (oral communication) that the following independent condition is also necessary:

$$(\exists \alpha \in (0, \pi/2)) : \sum_{b_k \in E \bigcap \{z : |\arg z - \pi| < \alpha\}} \text{Re} \frac{1}{x - b_k} \to 0, \quad x \to -\infty. \tag{0.10}$$

At the moment we do not know whether or not the set of conditions: (0.4), (0.8), (0.9), (0.10) gives a complete characterization of zero sets of entire absolutely monotonic functions.

The main result of the paper is the following characterization of zero sets situated in the right half-plane:

Theorem 2 *Let $E = \{b_k\}_{k=1}^{\infty} \subset \{z : Rez \geq 0\}$ be a set without finite accumulation points. The set E is the zero set of an entire absolutely monotonic function iff the conditions (0.4) and (0.8) are satisfied.*

The necessity of these conditions is obvious. Note that in our case the conditions (0.9) and (0.10) are satisfied automatically.

1 Main steps of the proof of Theorem 2

The first step of the proof of Theorem 2 is that of Theorem 1. This proof is contained in Sec. 3.

The second step is the proof of the following theorem which will be proved on the base of Theorem 1 in Sec. 4.

Theorem 3 *Let $E = \{b_k\}_{k=1}^{\infty} \subset \{z : Rez \geq 0\}$ be a set without finite accumulation points and satisfying conditions (0.4) and (0.8). There exists an entire function $\psi_1(z)$ with zero set E representable by the absolutely convergent in C integral*

$$\psi_1(z) = \int_0^\infty e^{zx} p_1(x) dx \tag{1.1}$$

where p_1 is a real continuous on R^+ function positive on an interval $(0, x_1)$, $x_1 > 0$.

The third step is the proof of the following theorem in Sec. 5.

Theorem 4 *Let $\psi(z)$ be an entire function bounded in C_ω, $\forall \omega \in R$ and $\psi(u) > 0, u \in R$. There exists an entire absolutely monotonic function $\chi(z)$ without zeros such that the product $\psi \cdot \chi$ is representable by the absolutely convergent in C integral*

$$\psi_2(z) := \psi(z)\chi(z) = \int_0^\infty e^{zx} p_2(x) dx, \tag{1.2}$$

where p_2 is a real continuous on R^+ function positive on a half-ray $[x_2, +\infty)$, $x_2 > 0$.

Sec. 6 is devoted to proofs of two technical lemmas. The fourth step of the proof of Theorem 2 is that of the following theorem in Sec. 7.

Theorem 5 *Let $p : R^+ \to R$ be a continuous function such that*

(a) the following integral is absolutely convergent in C:

$$\psi(z) := \int_0^\infty e^{zx} p(x)dx; \tag{1.3}$$

(b) $\psi(u) > 0,\ \forall u \in R$;

(c) there exist numbers $x_1, x_2,\ 0 < x_1 \leq x_2 < \infty$ such that $p(x) > 0$ for $x \in (0, x_1) \bigcup (x_2, +\infty)$.

Then $(\exists \varepsilon_0 > 0)(\forall \varepsilon \in (0, \varepsilon_0])(\exists \lambda_0 > 0)(\forall \lambda \geq \lambda_0)$ the function

$$\psi_{\lambda,\varepsilon}(z) := \psi(z) \cdot \exp(\lambda e^{\varepsilon z}) \tag{1.4}$$

admits the representation

$$\psi_{\lambda,\varepsilon}(z) = \int_0^\infty e^{zx} w_{\lambda,\varepsilon}(x)dx \tag{1.5}$$

where $w_{\lambda,\varepsilon}(x) \geq 0$ for all $x \in R^+$.

Note that Theorem 5 is of the same character as the results of Diamond and Essén [4].

2 Deduction of Theorem 2 from Theorems 3, 4, 5

Let E be a set satisfying the conditions of Theorem 2. By Theorem 3 there exists an entire function $\psi_1(z)$ with the zero set E admitting the representation (1.1) where $p_1(x)$ is a real continuous function positive on an interval $(0, x_1),\ x_1 > 0$. Let us show that $\psi_1(z)$ satisfies the conditions of Theorem 4.

Evidently, $\psi_1(z)$ is an entire function bounded in $C_\omega,\ \forall \omega \in R$. Moreover, $\psi_1(u) \in R$ for $u \in R$. Since $E \bigcap R = \emptyset$, we have $\psi_1(u) \neq 0$ for $u \in R$. Since $p_1(x) > 0$ for $x \in (0, x_1)$, we have $\psi_1(u) > 0$ for all $u < 0$ with $|u|$ being large enough. Therefore $\psi_1(u) > 0,\ \forall u \in R$, and the conditions of Theorem 4 are satisfied.

By Theorem 4, there exists an entire absolutely monotonic function $\chi(z)$ without zeros admitting the representation (1.2) where $p_2(x)$ is a real continuous function positive for $x \geq x_2 > 0$. Since $\chi(u) > 0,\ \forall u \in R$, we have $\psi_2(u) > 0,\ \forall u \in R$. Since $\chi(z)$ does not vanish, the zero set of $\psi_2(z)$ coincides with E. Being an entire absolutely monotonic function, $\chi(z)$ admits the representation

$$\chi(z) = \int_0^\infty e^{zx} Q(dx) \tag{2.1}$$

where Q is a finite nonnegative Borel measure on R^+. Hence

$$p_2(x) = (p_1 * Q)(x) = \int_0^x p_1(x - t)Q(dt). \tag{2.2}$$

Hence $p_2(x) > 0$ for $x \in (0, x_1)$ because $p_1(x) > 0$ for $x \in (0, x_1)$. Therefore $p_2(x)$ satisfies the conditions of Theorem 5.

By Theorem 5, for some $\varepsilon > 0,\ \lambda > 0$, the function (1.4) admits the representation (1.5) where $w_{\lambda,\varepsilon}(x) \geq 0$ for all $x \in R^+$. Evidently, the zero set of $\psi_{\lambda,\varepsilon}(z)$ coincides with E.

3 Proof of Theorem 1

As it was mentioned before, the proof of necessity is trivial. The below proof of sufficiency is based on an idea of I.V. Ostrovskii.

3.1. Let E be a set satisfying the conditions of Theorem 1. Let

$$E_- := E\bigcap\{z : \mathrm{Re}z < 0\} = \{a_k\}_{k=1}^{\infty}. \tag{3.1}$$

There exists a sequence of positive $\delta_k \uparrow +\infty, k \uparrow +\infty$, such that

$$\sum_{k=1}^{\infty} \frac{|\mathrm{Re}a_k| \cdot \delta_k + \delta_k^2}{|a_k|^2 + 1} < \infty. \tag{3.2}$$

Note that (3.2) implies

$$\delta_k = o(|a_k|), \quad k \to \infty. \tag{3.3}$$

Set

$$B_1(z) := \prod_{k=1}^{\infty} \frac{1 - z/a_k}{1 - z/(\delta_k - \bar{a}_k)}. \tag{3.4}$$

By (3.2) the infinite product (3.4) converges and is a meromorphic function. We shall show that $B_1(z)$ is bounded in $C_\omega \setminus K_\omega$, $\forall \omega \in R$, where K_ω is a compact subset of $\{z : \mathrm{Re}z \geq 0\}$. We have

$$|B_1(z)|^2 = \prod_{k=1}^{\infty}\left(1 + \frac{\delta_k^2 - 2\mathrm{Re}a_k \cdot \delta_k}{|a_k|^2}\right)\prod_{k=1}^{\infty}\left|\frac{a_k - z}{\delta_k - \bar{a}_k - z}\right|^2 =: C_1 \prod_{k=1}^{\infty}\left|\frac{a_k - z}{\delta_k - \bar{a}_k - z}\right|^2, \tag{3.5}$$

where $C_1 > 0$ does not depend on z. For $\delta_k > 2\omega$

$$\left|\frac{a_k - z}{\delta_k - \bar{a}_k - z}\right| \leq 1, \quad \forall z \in C_\omega. \tag{3.6}$$

In particular, for $\omega = 0$ (3.6) holds for any $k \in \mathbf{N}$, hence

$$|B_1(z)|^2 \leq C_1, \quad \forall z \in \{\mathrm{Re}z \leq 0\}. \tag{3.7}$$

Since $\delta_k \uparrow +\infty$, (3.6) holds for all $\omega > 0$ and all sufficiently large $k \geq k_0(\omega)$. So, $B_1(z)$ is bounded in $C_\omega \setminus K_\omega$, $\forall \omega > 0$, where $K_\omega \subset \{z : \mathrm{Re}z \geq 0\}$ is a compact set including points $-\bar{a}_k + \delta_k,\ k = 1, \ldots k_0(\omega) - 1$.

3.2. Let

$$E_+ := E\bigcap\{z : \mathrm{Re}z \geq 0\} = \{b_k\}_{k=1}^{\infty}. \tag{3.8}$$

Without loss of generality, $0 \notin E_+$. From (0.8) it follows

$$\sum_{b_k \in C_\omega \bigcap E_+} \frac{1}{|b_k|^2} < \infty, \quad \forall \omega \in R. \tag{3.9}$$

Hence, for any $n \in \mathbf{N}$, there is $A_n > 0$ such that

$$\sum_{n-1 \leq \mathrm{Re}b_k \leq n;\ |\mathrm{Im}b_k| \geq A_n} \frac{n}{|b_k|^2} \leq \frac{1}{n^2}. \tag{3.10}$$

Set

$$\Pi_n := \{z : n - 1 \leq \mathrm{Re}z \leq n,\ |\mathrm{Im}z| \leq A_n\}, \tag{3.11}$$

$$\Omega := \bigcup_{n=1}^{\infty} \Pi_n, \quad \Lambda := \{z : \mathrm{Re}z \geq 0\} \setminus \Omega. \tag{3.12}$$

By (3.10), there exists a sequence $0 < \mu_k \uparrow \infty,\ k \uparrow \infty$, such that

$$\sum_{b_k \in \Lambda} \frac{\mathrm{Re} b_k \cdot \mu_k + \mu_k^2}{|b_k|^2} < \infty, \tag{3.13}$$

hence

$$\mu_k = o(|b_k|), \quad k \to +\infty. \tag{3.14}$$

Set

$$B_2(z) := \prod_{b_k \in \Lambda} \frac{1 - z/b_k}{1 - z/(b_k + \mu_k)}. \tag{3.15}$$

By (3.13) the infinite product converges and is a meromorphic function. By the reasons analogous to those related to $B_1(z)$ one can show that $B_2(z)$ is bounded in $C_\omega \setminus K'_\omega$, $\forall \omega \in R^+$, where K'_ω is a compact set in $\{z : \mathrm{Re} z \geq 0\}$.

3.3. Let $V(z)$ be an entire function with zero set being union of three sets: the set of all poles of $B_1(z)$, the set of all poles of $B_2(z)$, and the set $E \bigcap \Omega$. Let us consider the entire function

$$f_0(z) := B_1(z) B_2(z) V(z). \tag{3.16}$$

The zero set of $f_0(z)$ coincides with E. But $f_0(z)$ is not necessary bounded in C_ω, $\forall \omega \in R$.

Further we shall need the following theorem being a simple particular case of the well-known theorem of M.V. Keldysh (see [5]).

Theorem B *Let $\tau(x) > 0$ be a continuous non-decreasing function on R^+ such that $\tau(x) \uparrow +\infty,\ x \uparrow +\infty$. Let $g(z)$ be a function analytic in the closed domain*

$$G = C \setminus \{z : Rez > 0,\ |Imz| < \tau(Rez)\}. \tag{3.17}$$

Then there exists an entire function $\Phi(z)$ such that

$$|g(z) - \Phi(z)| \leq 1, \quad \forall z \in G. \tag{3.18}$$

Evidently, there exists a function $\tau(x)$ satisfying the conditions of Theorem B and such that the corresponding domain G is free of zeros of $V(z)$. Applying Theorem B to $g(z) = \log V(z)$, we get the entire function $\Phi(z)$ such that

$$|\log V(z) - \Phi(z)| \leq 1, \quad \forall z \in G. \tag{3.19}$$

Evidently,

$$f(z) := f_0(z) \exp(-\Phi(z)) = B_1(z) B_2(z) V(z) \exp(-\Phi(z)) \tag{3.20}$$

is an entire function bounded in C_ω, $\forall \omega$, with zero set E. □

4 Proof of Theorem 3

4.1. Let E be a set satisfying conditions of Theorem 3. Let us construct an entire function $f(z)$ bounded in C_ω, $\forall \omega \in R$, by means the method of Sec.3. Since $E \bigcap \{z : \mathrm{Re} z < 0\} = \emptyset$, the factor $B_1(z)$ will be absent in (3.20) and

$$f(z) = B_2(z) V(z) \exp(-\Phi(z)). \tag{4.1}$$

Let

$$\Lambda_+ := \Lambda \bigcap \{z : \mathrm{Im} z > 0\}. \tag{4.2}$$

Since E is symmetric with respect to R,

$$B_2(z) = \prod_{b_k \in \Lambda_+} \left(1 + \frac{2\mathrm{Re} b_k \cdot \mu_k + \mu_k^2}{|b_k|^2}\right) \cdot \prod_{b_k \in \Lambda_+} \frac{(b_k - z)(\bar{b}_k - z)}{(b_k + \mu_k - z)(\bar{b}_k + \mu_k - z)}. \tag{4.3}$$

From (3.13) and (3.14) it follows that there exists $k_0 \in \mathbf{N}$ such that

$$\frac{\mathrm{Re} b_k \cdot \mu_k + \mu_k^2}{|b_k|^2} \leq \frac{1}{8}, \quad \frac{\mu_k}{|b_k|} \leq \frac{1}{4}, \quad \forall k \geq k_0. \tag{4.4}$$

Without loss of generality we can assume that (4.4) holds for all $k \in \mathbf{N}$ and

$$\mathrm{Re} b_k \leq \frac{1}{4}|\mathrm{Im} b_k|, \quad b_k \in \Lambda. \tag{4.5}$$

(We enlarge the region Ω if necessary.)

In what follows we shall need estimations of $\log B_2(-r)$ and its derivatives. To write them, we introduce some notations.

Let us fix $0 < \beta < 1$ and set

$$q(r) := \frac{1}{r^{1+\beta}} + \sum_{b_k \in \Lambda_+} \frac{2r\mu_k}{(|b_k|^2 + r^2)^2} + \sum_{b_k \in \Lambda_+} \frac{\mu_k}{|b_k|} \cdot \frac{1}{|b_k|^2 + r^2}. \tag{4.6}$$

By (3.13) and (3.14) both series in the right-hand side of (4.6) converge uniformly with respect to r on each compact subset of R^+ and $q(r) \to 0$, as $r \to \infty$. Let

$$\begin{aligned} Q(r) \quad := \quad & \int_r^\infty q(t)dt = \\ & \frac{1}{\beta} \cdot \frac{1}{r^\beta} + \sum_{b_k \in \Lambda_+} \frac{\mu_k}{|b_k|^2 + r^2} + \sum_{b_k \in \Lambda_+} \frac{\mu_k}{|b_k|^2} \left(\frac{\pi}{2} - \arctan \frac{r}{|b_k|} \right), \quad r \geq 1. \end{aligned} \tag{4.7}$$

Note that $Q(r) \downarrow 0$, as $r \uparrow \infty$. Further we shall denote by C with indices positive constants.

Lemma 1 *The following estimations hold:*

$$\begin{aligned} \log B_2(-r) \quad &\geq \quad -C_2 r Q(r), \quad r \geq 1; & (4.8) \\ |(\log B_2(z))'|_{z=-r} \quad &\leq \quad C_3 Q(r), \quad r \geq 1; & (4.9) \\ \left| \left(\frac{d^j}{dz^j} \log B_2(z) \right) \right|_{z=-r} \quad &\leq \quad C_4 \cdot j! q(r) r^{2-j}, \quad j = 2, 3, \dots, \ r \geq 1; & (4.10) \\ \log(|B_2(-r+iy)|/B_2(-r)) \quad &\leq \quad C_5 q(r) y^2, \quad y \in R, \ r \geq 1; & (4.11) \\ \log(|B_2(-r+iy)|/B_2(-r)) \quad &\leq \quad C_6 q(y) y^2, \quad y \in R^+, \ 1 \leq r \leq y/2. & (4.12) \end{aligned}$$

The proof of the lemma will be given in Sec. 6.

4.2. Theorem 3 is an immediate corollary of the following result.

Theorem 3$'$. *Let $f(z)$ be the function defined by (4.1). There exists an entire function $\varphi(z)$ without zeros such that the product $\psi_1(z) := f(z)\varphi(z)$ is representable in the form*

$$\psi_1(z) = \int_0^\infty e^{zx} p_1(x) dx, \tag{4.13}$$

where p_1 is a real continuous function on R^+ positive on some interval $(0, x_1)$, $x_1 > 0$.

Let

$$\begin{aligned} \Delta(t) \quad := \quad & \left(-\frac{q(t)}{t} \right)' t^2 = \frac{2+\beta}{t^{1+\beta}} + \sum_{b_k \in \Lambda_+} \frac{8\mu_k t^3}{(t^2 + |b_k|^2)^3} + \\ & \sum_{b_k \in \Lambda_+} \frac{\mu_k}{|b_k|} \cdot \frac{1}{t^2 + |b_k|^2} + \sum_{b_k \in \Lambda_+} \frac{\mu_k}{|b_k|} \cdot \frac{2t^2}{(t^2 + |b_k|^2)^2}, \quad t \geq 1. \end{aligned} \tag{4.14}$$

Lemma 2 *The function $\Delta(t)$ posesses the following properties:*

(a) $\Delta(t) > 0, \quad t \geq 1$*;*

(b) $\int^{\infty} \Delta(t)dt < \infty$*;*

(c) $\Delta(t)t \to 0,$ *as* $t \to +\infty$*;*

(d) $\Delta(t)t^3 \uparrow +\infty,$ *as* $t \uparrow +\infty$*;*

(e) $\Delta(t) \leq 4q(t), \quad t \geq 1.$

Proof of Lemma 2 is obvious.

Set

$$h_1(z) := \int_0^A (e^{tz} - 1)\frac{\Delta(1/t)}{t^3}dt, \quad 0 < A \leq 1. \tag{4.15}$$

Since Lemma 2 (b), the integral in the right-hand side of (4.15) is absolutely convergent and $h_1(z)$ is an entire function. Since

$$h_1^{(k)}(x) > 0, \quad \forall k \in \mathbf{N}\bigcup\{0\},\ \forall x \in R, \tag{4.16}$$

the function $h_1(z)$ is entire absolutely monotonic. It is easy to see that

$$\mathrm{Re} h_1(x + iy) \leq h_1(x), \quad \forall x, y \in R. \tag{4.17}$$

Let $\varphi_\alpha(z),\ \alpha \in (0, 1)$, be the entire function defined by

$$\varphi_\alpha(z) := \exp \int_0^1 (e^{zt} - 1)t^{-1-\alpha}dt. \tag{4.18}$$

Note that $\log \varphi_\alpha(z)$ is a particular case of $h_1(z)$ corresponding to $\Delta = \Delta_\alpha = t^{\alpha-2},\ A = 1$.

Lemma 3 *For a fixed ω, the following asymptotic equality holds in the half-plane C_ω:*

$$\log \varphi_\alpha(z) = -C_\alpha|z|^\alpha e^{i\alpha(\arg z - \pi)} + O(1), \tag{4.19}$$

$$\pi/2 < \arg z < 3\pi/2, \quad |z| \to \infty, \tag{4.20}$$

where $C_\alpha > 0$ does not depend on z.

Proof of Lemma 3. Setting $z = \omega + \rho e^{i\theta},\ \pi/2 < \theta < 3\pi/2$, we obtain (4.19) from the following calculations:

$$\log \varphi_\alpha(z) = \rho^\alpha \int_0^\rho (e^{\omega t/\rho} \exp(e^{i\theta}t) - 1)t^{-1-\alpha}dt = \rho^\alpha \int_0^\infty (\exp(e^{i\theta}t) - 1)t^{-1-\alpha}dt -$$
$$\rho^\alpha \int_\rho^\infty (\exp(e^{i\theta}t) - 1)t^{-1-\alpha}dt + \rho^\alpha \int_0^\rho \exp(e^{i\theta}t)(e^{\omega t/\rho} - 1)t^{-1-\alpha}dt =$$
$$\rho^\alpha e^{-\alpha(\pi-\theta)i} \int_0^\infty (e^{-s} - 1)s^{-1-\alpha}ds + O(1), \quad \rho \to \infty. \tag{4.21}$$

□

4.3. Set

$$\begin{aligned}\psi_1(z) \quad &:= \quad f(z)\exp[M(h_1(z) - h_1(0))]\varphi_\alpha(z) = \\ &\qquad B_2(z)V(z)e^{-\Phi(z)}\exp[M(h_1(z) - h_1(0))]\varphi_\alpha(z),\end{aligned} \tag{4.22}$$

where the constant $M > 0$ will be choosen later. Let

$$\varphi(z) := \exp[M(h_1(z) - h_1(0))]\varphi_\alpha(z). \tag{4.23}$$

Evidently, $\varphi(z)$ is an entire absolutely monotonic function. Let

$$p_1(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\eta}\psi_1(i\eta)d\eta, \quad \eta \in R. \tag{4.24}$$

Taking into account boundedness of $f(z)$ in C_ω, $\forall\omega \in R$, and (4.17), (4.19), we see that the integral in the right-hand side of (4.24) converges absolutely and uniformly with respect to $x \in R$. Using (4.17), (4.19), we can transfer the integration in (4.24) to the line $\{z : \mathrm{Im} z = \xi\}$, $\forall\xi \in R$. Noting that

$$(\psi_1(x) \in R,\ \forall x \in R) \Rightarrow (\psi_1(\xi + i\eta) = \overline{\psi_1(\xi - i\eta)},\ \xi,\eta \in R), \tag{4.25}$$

we get

$$p_1(x) = \frac{e^{-x\xi}\psi_1(\xi)}{2\pi}\int_0^{\infty} \mathrm{Re}\left(e^{-ix\eta}\frac{\psi_1(\xi+i\eta)}{\psi_1(\xi)}\right)d\eta, \quad \forall\xi \in R. \tag{4.26}$$

Hence $p_1(x) \in R$ for $x \in R$ and

$$\mathrm{sign} p_1(x) = \mathrm{sign}\int_0^{\infty} \mathrm{Re}\left(e^{-ix\eta}\frac{\psi_1(\xi+i\eta)}{\psi_1(\xi)}\right)d\eta, \quad \forall\xi \in R. \tag{4.27}$$

Using (4.17) and (4.19) and putting $\xi \to -\infty$ in (4.26), we conclude that $p_1(x) = 0$ for $x < 0$. Taking sufficiently large positive ξ in (4.26), we get

$$p_1(x) = O(e^{-Cx}), \quad x \to +\infty,\ \forall C > 0. \tag{4.28}$$

Hence, by the Fourier inversion formula,

$$\psi_1(z) = \int_0^{\infty} e^{xz}p_1(x)dx, \quad \forall z \in C. \tag{4.29}$$

We are going to show that $p_1(x) > 0$ on some interval $(0, x_1)$, $x_1 > 0$. For this, we represent the integral in the right-hand side of (4.27) in the form

$$\left(\int_0^{\varepsilon_1} + \int_{\varepsilon_1}^{\varepsilon_2} + \int_{\varepsilon_2}^{\infty}\right)\mathrm{Re}\left(e^{-ix\eta}\frac{\psi_1(\xi+i\eta)}{\psi_1(\xi)}\right)d\eta =: I_1 + I_2 + I_3, \tag{4.30}$$

where $\varepsilon_1 = \varepsilon_1(\xi)$, $\varepsilon_2 = \varepsilon_2(\xi)$ will be choosen later.

We shall estimate I_1 from below, $|I_2|$ and $|I_3|$ from above and are going to show that $I_1 > 0$ and $I_1 > |I_2| + |I_3|$ for $|\xi|$ being large enough.

4.4.

Lemma 4 *Let*

$$\theta(r) := q(r) + \int_r^{\infty}\frac{q(u)}{u}du. \tag{4.31}$$

The following estimations hold:

$$h_1'(\xi) \geq \frac{2}{e}Q(|\xi|); \tag{4.32}$$

$$h_1''(\xi) \geq \frac{1}{e}q(|\xi|); \tag{4.33}$$

$$h_1'(\xi) \leq \int_{|\xi|}^{\infty}\Delta(u)du + C_8\Delta(|\xi|)|\xi|; \tag{4.34}$$

$$0 < h_1^{(j)}(\xi) \leq C_9\frac{j!}{|\xi|^{j-2}}\theta(|\xi|), \quad j = 2, 3, \ldots; \tag{4.35}$$

$$(\log\varphi_\alpha(\xi))' \geq C_{10}|\xi|^{\alpha-1}; \tag{4.36}$$

$$(\log\varphi_\alpha(\xi))'' \geq C_{11}|\xi|^{\alpha-2}; \tag{4.37}$$

$$0 < \frac{d^j}{d\xi^j}\log\varphi_\alpha(\xi) \leq C_{12}\, j!|\xi|^{\alpha-j}, \quad j = 1, 2, \ldots, \tag{4.38}$$

where $\xi \leq \xi_0 < 0$.

Proof of Lemma 4. Differentiating (4.15), we get for $|\xi| > 1/A$:

$$h_1^{'}(\xi) \geq \int_0^{1/|\xi|} e^{-t|\xi|} \frac{\Delta(1/t)}{t^2} dt \geq \frac{1}{e} \int_{|\xi|}^{\infty} \Delta(u) du. \tag{4.39}$$

Using (4.14) and (4.7), we obtain (4.32).

The validity of (4.33) follows from the calculation:

$$\begin{aligned} h_1^{''}(\xi) &= \int_0^A e^{-t|\xi|} \frac{\Delta(1/t)}{t} dt \geq \int_0^{1/|\xi|} e^{-t|\xi|} \frac{\Delta(1/t)}{t} dt \geq \frac{1}{e} \int_{|\xi|}^{\infty} \frac{\Delta(u)}{u} du = \\ & \frac{1}{e} \int_{|\xi|}^{\infty} \left(-\frac{q(u)}{u}\right)' u du = \frac{1}{e} \theta(|\xi|) \geq \frac{1}{e} q(|\xi|). \end{aligned} \tag{4.40}$$

By (4.15)

$$\begin{aligned} h_1^{(j)}(\xi) &= \int_0^A e^{-t|\xi|} \Delta(1/t) t^{j-3} dt = \left(\int_0^{1/|\xi|} + \int_{1/|\xi|}^A \right) e^{-t|\xi|} \Delta(1/t) t^{j-3} dt =: \\ & L_j^1(\xi) + L_j^2(\xi), \quad j = 1, 2, \ldots. \end{aligned} \tag{4.41}$$

As it was mentioned in (4.16) , $h_1^{(j)}(\xi) > 0,\ j = 1, 2, \ldots.$ Further,

$$L_j^1(\xi) \leq \int_0^{1/|\xi|} \Delta(1/t) t^{j-3} dt = \int_{|\xi|}^{\infty} \Delta(u) u^{1-j} du, \quad j = 1, 2, \ldots, \tag{4.42}$$

$$L_1^1(\xi) \leq \int_{|\xi|}^{\infty} \Delta(u) du, \tag{4.43}$$

$$L_j^1(\xi) \leq \int_{|\xi|}^{\infty} \frac{\Delta(u)}{u} \frac{du}{u^{j-2}} \leq \frac{1}{|\xi|^{j-2}} \int_{|\xi|}^{\infty} \frac{\Delta(u)}{u} du, \quad j = 2, 3, \ldots. \tag{4.44}$$

Integrating (4.44) by parts as in (4.40), we get

$$L_j^1(\xi) \leq \frac{1}{|\xi|^{j-2}} \theta(|\xi|), \quad j = 2, 3, \ldots. \tag{4.45}$$

Using Lemma 2 (d), we obtain

$$\begin{aligned} L_j^2(\xi) &= \int_{1/|\xi|}^A e^{-t|\xi|} t^j \frac{\Delta(1/t)}{t^3} dt \leq \Delta(|\xi|) |\xi|^3 \int_{1/|\xi|}^A e^{-t|\xi|} t^j dt \leq \\ &\leq \Delta(|\xi|) \frac{j!}{|\xi|^{j-2}}, \quad j = 1, 2, \ldots. \end{aligned} \tag{4.46}$$

For $j = 1$ we derive (4.34) from (4.43) and (4.46). Further, for $j = 2, 3, \ldots$, (4.45) and (4.46) yield

$$h_1^{(j)}(\xi) \leq \frac{j!}{|\xi|^{j-2}} (\theta(|\xi|) + \Delta(|\xi|)). \tag{4.47}$$

Using Lemma 2 (e) and the definition of $\theta(r)$, we get the right-hand side of (4.35).

Since $\log \varphi_\alpha(z)$ is a particular case of $h_1(z)$ (for $\Delta(u) = \Delta_\alpha(u) = u^{\alpha-2},\ A = 1$), the estimations (4.36), (4.37), (4.38) follow. □

4.5. Since the function $\log(V(\xi + z)e^{-\Phi(\xi+z)})$ is analytic in the disc $\{z : |z| < |\xi|/2\}$ for $\xi < 0$ and (3.19) holds, Caushy's inequality implies

$$\left| \frac{d^j}{d\xi^j} \log(V(\xi + z) e^{-\Phi(\xi+z)}) \right| \leq j! \frac{2^j}{|\xi|^j}. \tag{4.48}$$

Set

$$b_1(\xi) := \log \psi_1(\xi) = \log B_2(\xi) + \log \varphi_\alpha(\xi) + M(h_1(\xi) - h_1(0)) + \log(V(\xi)e^{-\Phi(\xi)}). \tag{4.49}$$

Since $f(\xi + z) \neq 0$ for $|z| < |\xi|$, $\xi < 0$, we have for $|\eta| < |\xi|/2$

$$\log \left\{ e^{-ix\eta} \frac{\psi_1(\xi + i\eta)}{\psi_1(\xi)} \right\} = -ix\eta + \sum_{j=1}^{\infty} \frac{(i\eta)^j}{j!} b_1^{(j)}(\xi). \tag{4.50}$$

By (4.9),(4.32), (4.36) and (4.48),

$$\begin{aligned} b_1^{'}(\xi) &= (\log B_2(\xi))^{'} + (\log \varphi_\alpha(\xi))^{'} + Mh_1^{'}(\xi) + \left[\log\left(V(\xi)e^{-\Phi(\xi)}\right)\right]^{'} \geq \\ & -C_3 Q(|\xi|) + C_{10}|\xi|^{\alpha-1} + \frac{2M}{e} Q(|\xi|) - \frac{2}{|\xi|}. \end{aligned} \tag{4.51}$$

Further we shall assume $M \geq eC_3$. Then, for $|\xi| > 1$,

$$b_1^{'}(\xi) \geq C_3 Q(|\xi|) + C_{14}|\xi|^{\alpha-1} > 0. \tag{4.52}$$

On the other hand, (4.9), (4.34), (4.38) and (4.48) imply

$$b_1^{'}(\xi) \leq C_3 Q(|\xi|) + C_{12}|\xi|^{\alpha-1} + \int_{|\xi|}^{\infty} \Delta(u)du + C_8 \Delta(|\xi|)|\xi| + \frac{2}{|\xi|}. \tag{4.53}$$

Taking into account Lemma 2 (b) and (c), we conclude that

$$b_1^{'}(\xi) \to 0, \quad \text{as} \quad \xi \to -\infty. \tag{4.54}$$

Moreover, (4.10), (4.33), (4.37) and (4.48) imply

$$\begin{aligned} b_1^{''}(\xi) &= (\log B_2(\xi))^{''} + (\log \varphi_\alpha(\xi))^{''} + Mh_1^{''}(\xi) + \left[\log\left(V(\xi)e^{-\Phi(\xi)}\right)\right]^{''} \geq \\ & -2C_4 q(|\xi|) + C_{11}|\xi|^{\alpha-2} + \frac{M}{e} q(|\xi|) - \frac{8}{|\xi|^2}. \end{aligned} \tag{4.55}$$

Assuming $M \geq 4C_4 e$, we shall have

$$b_1^{''}(\xi) \geq 2C_4 q(|\xi|) + C_{15}|\xi|^{\alpha-2}, \quad |\xi| \geq 1. \tag{4.56}$$

From (4.52), (4.54), (4.56) we conclude that $b_1^{'}(\xi) \downarrow 0$ as $\xi \downarrow -\infty$. Therefore the equation

$$b_1^{'}(\xi) = x \tag{4.57}$$

has a unique solution $\xi(x)$ for every x, $0 \leq x \leq x_0$, such that

$$\xi(x) \downarrow -\infty, \quad \text{as} \quad x \downarrow 0. \tag{4.58}$$

Substituting $\xi = \xi(x)$ into (4.50), we get

$$\log \left\{ e^{-ix\eta} \frac{\psi_1(\xi + i\eta)}{\psi_1(\xi)} \right\} = -\frac{b_1^{''}(\xi)}{2}\eta^2 + \tau_1(\xi, \eta), \tag{4.59}$$

where

$$\tau_1(\xi, \eta) := \sum_{j=3}^{\infty} \frac{(i\eta)^j}{j!} b_1^{(j)}(\xi). \tag{4.60}$$

By (4.10), (4.38), (4.35) and (4.48),

$$|b_1^{(j)}(\xi)| \le |(\log B_2(\xi))^{(j)}| + (\log \varphi_\alpha(\xi))^{(j)} + M h_1^{(j)}(\xi) + \left| \left[\log(V(\xi) e^{-\Phi(\xi)}) \right]^{(j)} \right| \le \tag{4.61}$$

$$C_4 j! q(|\xi|)|\xi|^{2-j} + C_{12} j! |\xi|^{\alpha-j} + C_{10} j! |\xi|^{2-j} \theta(|\xi|) + j! \frac{2^j}{|\xi|^j}, \tag{4.62}$$

whence, using the definition of $\theta(r)$, we get

$$|b_1^{(j)}(\xi)| \le C_{16} j! (\theta(|\xi|)|\xi|^2 + |\xi|^\alpha) \left(\frac{2}{|\xi|} \right)^j, \quad j = 2, 3, \ldots. \tag{4.63}$$

For $|\eta| \le |\xi|/4$, (4.63) implies

$$|\tau_1(\xi, \eta)| \le C_{16}(\theta(|\xi|)|\xi|^2 + |\xi|^\alpha) \sum_{j=3}^{\infty} \left(\frac{2|\eta|}{|\xi|} \right)^j \le C_{17} \frac{\theta(|\xi|) + |\xi|^{\alpha-2}}{|\xi|} |\eta|^3. \tag{4.64}$$

We choose

$$\varepsilon_1 = \varepsilon_1(\xi) := \left(\frac{\pi}{3C_{17}} \cdot \frac{|\xi|}{\theta(|\xi|) + |\xi|^{\alpha-2}} \right)^{1/3}. \tag{4.65}$$

Then, for $|\eta| < \varepsilon_1$, the inequality holds

$$|\tau_1(\xi, \eta)| \le \frac{\pi}{3}. \tag{4.66}$$

Using this and (4.59), we obtain

$$I_1 = \int_0^{\varepsilon_1} \exp\left\{ -\frac{b_1''(\xi)}{2} \eta^2 + \mathrm{Re}\tau_1(\xi, \eta) \right\} \cos(\mathrm{Im}\tau_1(\xi, \eta)) d\eta \ge \frac{1}{2} e^{-\pi/3} \int_0^{\varepsilon_1} \exp\left(-\frac{b_1''(\xi)}{2} \eta^2 \right) d\eta. \tag{4.67}$$

Hence, by (4.63),

$$I_1 \ge \frac{1}{2} e^{-\pi/3} (\theta(|\xi|) + |\xi|^{\alpha-2})^{-1/2} \int_0^{(\theta(|\xi|)+|\xi|^{\alpha-2})^{1/2}\varepsilon_1} \exp(-C_{18} u^2) du. \tag{4.68}$$

Note that (4.65) implies

$$(\theta(|\xi|) + |\xi|^{\alpha-2})^{1/2} \varepsilon_1 = C_{19} |\xi|^{1/3} (\theta(|\xi|) + |\xi|^{\alpha-2})^{1/6} \ge C_{19} |\xi|^{\alpha/6} \to \infty, \quad \text{as} \quad \xi \to -\infty. \tag{4.69}$$

Thus, (4.68) implies

$$I_1 \ge C_{20} (\theta(|\xi|) + |\xi|^{\alpha-2})^{-1/2} \to \infty, \quad \text{as} \quad \xi \to -\infty. \tag{4.70}$$

4.6. Set

$$\varepsilon_2 = \varepsilon_2(\xi) = 2|\xi|. \tag{4.71}$$

Evidently,

$$\varepsilon_1(\xi) = O(|\xi|^{1-\alpha/3}) < \varepsilon_2(\xi) \tag{4.72}$$

for sufficiently large $|\xi|$. We have

$$|I_2| \le \int_{\varepsilon_1}^{\varepsilon_2} \left| \frac{B_2(\xi + i\eta)}{B_2(\xi)} \right| \cdot \left| \frac{\varphi_\alpha(\xi + i\eta)}{\varphi_\alpha(\xi)} \right| \cdot \left| \frac{V(\xi + i\eta) e^{-\Phi(\xi + i\eta)}}{V(\xi) e^{-\Phi(\xi)}} \right| \exp\{M(\mathrm{Re} h_1(\xi + i\eta) - h_1(\xi))\} d\eta. \tag{4.73}$$

For sufficiently large $|\xi|$,

$$\mathrm{Re} h_1(\xi + i\eta) - h_1(\xi) = -2 \int_0^A e^{-t|\xi|} \sin^2 \frac{t\eta}{2} \Delta(\frac{1}{t}) \frac{dt}{t^3} \le -2 \int_0^{1/|\xi|} e^{-t|\xi|} \sin^2 \frac{t\eta}{2} \Delta(\frac{1}{t}) \frac{dt}{t^3}. \tag{4.74}$$

Since $t\eta/2 \le \eta/(2|\xi|) \le 1 < \pi/2$ for $\eta \le 2|\xi|$, we have

$$\mathrm{Re}h_1(\xi+i\eta) - h_1(\xi) \le -\frac{2}{e\pi^2}\eta^2 \int_0^{1/|\xi|} \Delta(\frac{1}{t})\frac{dt}{t}. \tag{4.75}$$

Integrating by parts as in (4.40), we get

$$\mathrm{Re}h_1(\xi+i\eta) - h_1(\xi) \le -\frac{2}{e\pi^2}\eta^2\theta(|\xi|). \tag{4.76}$$

Substituting $\Delta = \Delta_\alpha = u^{\alpha-2}$ into (4.75), we obtain

$$\log|\varphi_\alpha(\xi+i\eta)| - \log\varphi_\alpha(\xi) \le -\frac{2}{e\pi^2}\eta^2 \int_0^{1/|\xi|} t^{1-\alpha}dt = -\frac{2}{e\pi^2(2-\alpha)}\eta^2|\xi|^{\alpha-2}. \tag{4.77}$$

From (4.11), (3.19), (4.76), (4.77), we derive

$$\begin{aligned}\log|\psi_1(\xi+i\eta)| - \log\psi_1(\xi) = (\log|B_2(\xi+i\eta)| - \log B_2(\xi)) + (\log|\varphi_\alpha(\xi+i\eta)| - \\ \log\varphi_\alpha(\xi)) + M(\mathrm{Re}h_1(\xi+i\eta) - h_1(\xi)) + \left[\log\left|V(\xi+i\eta)e^{-\Phi(\xi+i\eta)}\right| - \log\left(V(\xi)e^{-\Phi(\xi)}\right)\right] \le \\ C_5 q(|\xi|)\eta^2 - \frac{2}{e\pi^2(2-\alpha)}\eta^2|\xi|^{\alpha-2} - \frac{2}{e\pi^2}M\eta^2\theta(|\xi|) + C_{21}.\end{aligned} \tag{4.78}$$

Assuming $M \ge C_5 e\pi^2$, we get

$$\log|\psi_1(\xi+i\eta)| - \log\psi_1(\xi) \le -C_{22}(\theta(|\xi|) + |\xi|^{\alpha-2})\eta^2 + C_{21}. \tag{4.79}$$

Since $\eta \ge \varepsilon_1$, (4.65) implies

$$\begin{aligned}\log|\psi_1(\xi+i\eta)| - \log\psi_1(\xi) \le -C_{22}(\theta(|\xi|) + |\xi|^{\alpha-2})\varepsilon_1^2 + C_{21} \le \\ -C_{23}|\xi|^{2/3}(\theta(|\xi|) + |\xi|^{\alpha-2})^{1/3} + C_{21} = -C_{23}(\theta(|\xi|)|\xi|^2 + |\xi|^\alpha)^{1/3} + C_{21}.\end{aligned} \tag{4.80}$$

Hence

$$|I_2| \le 2e^{C_{21}}|\xi| \exp[-C_{23}(\theta(|\xi|)|\xi|^2 + |\xi|^\alpha)^{1/3}] \to 0, \quad \text{as} \quad \xi \to -\infty. \tag{4.81}$$

4.7. We have

$$|I_3| \le \int_{\varepsilon_2}^\infty \left|\frac{B_2(\xi+i\eta)}{B_2(\xi)}\right| \cdot \left|\frac{\varphi_\alpha(\xi+i\eta)}{\varphi_\alpha(\xi)}\right| \cdot \left|\frac{V(\xi+i\eta)e^{-\Phi(\xi+i\eta)}}{V(\xi)e^{-\Phi(\xi)}}\right| \exp\{M(\mathrm{Re}h_1(\xi+i\eta) - h_1(\xi))\}d\eta. \tag{4.82}$$

For $|\xi| > 1/(2A)$, we have $\eta \ge 2|\xi| \ge 1/A$, therefore

$$\mathrm{Re}h_1(\xi+i\eta) - h_1(\xi) = -2\int_0^A e^{-t|\xi|}\sin^2\frac{t\eta}{2}\Delta(\frac{1}{t})\frac{dt}{t} \le -2\int_0^{1/\eta} e^{-t|\xi|}\sin^2\frac{t\eta}{2}\Delta(\frac{1}{t})\frac{dt}{t^3}. \tag{4.83}$$

Since $t\eta/2 \le 1/2 < \pi/2$ for $0 \le t \le 1/2$,

$$\mathrm{Re}h_1(\xi+i\eta) - h_1(\xi) \le -\frac{2\eta^2}{\pi^2}e^{-|\xi|/\eta}\int_0^{1/\eta}\Delta(\frac{1}{t})\frac{dt}{t} \le -\frac{2\eta^2}{\pi^2\sqrt{e}}\int_\eta^\infty \Delta(u)\frac{du}{u}. \tag{4.84}$$

Integrating by parts as in (4.40), we obtain

$$\mathrm{Re}h_1(\xi+i\eta) - h_1(\xi) \le -\frac{2}{\pi^2\sqrt{e}}\theta(\eta)\eta^2. \tag{4.85}$$

Since $|\xi| < \eta/2$, we derive from (4.12)

$$\log|B_2(\xi+i\eta)| - \log B_2(\xi) \le C_6 q(\eta)\eta^2 \le C_6\theta(\eta)\eta^2, \quad 1 \le |\xi| < \frac{\eta}{2}. \tag{4.86}$$

Using (4.82), (4.85), (3.19) and the inequality

$$|\varphi_\alpha(\xi+i\eta)| \le \varphi_\alpha(\xi) \tag{4.87}$$

(which is a particular case of (4.17)), we get

$$\left|\frac{\psi_1(\xi+i\eta)}{\psi_1(\xi)}\right| = \left|\frac{B_2(\xi+i\eta)}{B_2(\xi)}\right| \cdot \left|\frac{\varphi_\alpha(\xi+i\eta)}{\varphi_\alpha(\xi)}\right| \cdot \left|\frac{V(\xi+i\eta)e^{-\Phi(\xi+i\eta)}}{V(\xi)e^{-\Phi(\xi)}}\right| \cdot \tag{4.88}$$

$$\exp\{M(\mathrm{Re}h_1(\xi+i\eta)-h_1(\xi))\} \le C_{24}\exp\left\{C_6\theta(\eta)\eta^2 - \frac{2M}{\pi^2\sqrt{e}}\theta(\eta)\eta^2\right\}, \quad 2|\xi|<\eta<\infty.$$

Assuming $M > \pi^2 C_6\sqrt{e}$ and using (4.31), (4.6), we obtain

$$\left|\frac{\psi_1(\xi+i\eta)}{\psi_1(\xi)}\right| \le C_{24}\exp(-C_6\theta(\eta)\eta^2) \le C_{24}\exp(-C_6 q(\eta)\eta^2) \le C_{24}\exp(-C_6\eta^{1-\beta}). \tag{4.89}$$

Thus,

$$|I_3| \le C_{24}\int_{2|\xi|}^{\infty} \exp(-C_6\eta^{1-\beta})d\eta \to 0, \quad \text{as} \quad |\xi|\to\infty. \tag{4.90}$$

Substituting (4.70), (4.81), (4.90) into (4.30), we conclude that

$$\int_0^\infty \mathrm{Re}\left(e^{-ix\eta}\frac{\psi_1(\xi+i\eta)}{\psi_1(\xi)}\right)d\eta > 0, \quad \text{for} \quad \xi \le \xi_0 < 0. \tag{4.91}$$

Hence (4.27) and (4.58) imply that $p_1(x) > 0$ for $0 < x < x_0$.

5 Proof of Theorem 4

5.1.

Lemma 5 *Let $g(r) \ge 1$ be a continuous nondecreasing function on R^+, let $\beta > 0$. There exists an entire function of the form*

$$h(z) := \sum_{m=0}^{\infty} a_m \varphi_\alpha^m(z), \quad a_m \ge 0, \tag{5.1}$$

where $\varphi_\alpha(z)$ is defined by (4.18), such that:

(a) $h(r) \ge g(r), \quad \forall r \ge 0$;

(b) $h(r + h^{-\beta}(r)) \le 2h(r), \quad \forall r \ge r_0$.

We postpone the proof to Sec. 6.

5.2. Let $\psi(z)$ be a function satisfying the conditions of Theorem 4. Set

$$F(z) := \psi(z)\varphi_\alpha(z). \tag{5.2}$$

By Lemma 3

$$|F(z)| \le K_\omega^2 \exp(-K_\omega^3|z|^\alpha), \quad \forall z \in C_\omega,\ \forall \omega \in R, \tag{5.3}$$

holds, where K_ω^2 and K_ω^3 are positive constants not depending on z.

Since $F(x) > 0,\ \forall x \in R$, there exists a continuous function $\delta(r) > 0$ on R^+ such that $F(z)$ does not vanish in $\{z : |z-\xi| < \delta(\xi)\}$. Let

$$g(r) := \max\Bigg\{1+r^2;\ \max_{0\le\xi\le r}(\delta(\xi))^{-8};\ \max_{0\le\xi\le r}\left|(\log F(\xi))''\right|;\ 3|\log\left(\max_{0\le\xi\le r,\,\eta\in R}|F(\xi+i\eta)|\right)|;$$
$$3\max_{0\le\xi\le r}|(\log F(\xi))'|;\ \sup_{0\le\xi\le r}\int_{-\infty}^{\infty}\left|\frac{F(\xi+i\eta)}{F(\xi)}\right|d\eta\Bigg\}. \qquad (5.4)$$

This choice of $g(r)$ will be justified in process of proof.

Taking this $g(r)$, $\beta = 1/8$, let us consider the entire function of Lemma 5. Set

$$\psi_2(z) := F(z)\exp h(z) = \psi(z)\varphi_\alpha(z)\exp h(z), \qquad (5.5)$$
$$\chi(z) := \varphi_\alpha(z)\exp h(z). \qquad (5.6)$$

Evidently, $\chi(z)$ is an entire absolutely monotonic function. Let

$$p_2(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\eta}\psi_2(i\eta)d\eta, \quad \eta \in R. \qquad (5.7)$$

Using reasonings analogous to those of Sec. 4, we show that

$$p_2(x) = 0 \quad \text{for} \quad x < 0; \qquad p_2(x) = O(e^{-Cx}),\ x \to +\infty,\ \forall C > 0. \qquad (5.8)$$

By the Fourier inversion formula,

$$\psi_2(z) = \int_0^\infty e^{xz}p_2(x)dx. \qquad (5.9)$$

As in Sec. 4, the integration can be transferred to any line $\{z : \mathrm{Im} z = \xi\}$, and we get

$$p_2(x) = \frac{e^{-x\xi}\psi_2(\xi)}{2\pi}\int_0^\infty \mathrm{Re}\left(e^{-ix\xi}\frac{\psi_2(\xi+i\eta)}{\psi_2(\xi)}\right)d\eta, \qquad (5.10)$$
$$\mathrm{sign} p_2(x) = \mathrm{sign}\int_0^\infty \mathrm{Re}\left(e^{-ix\eta}\frac{\psi_2(\xi+i\eta)}{\psi_2(\xi)}\right)d\eta. \qquad (5.11)$$

We are going to show that $p_2(x) > 0$ on some interval $(x_2,\infty),\ x_2 > 0$. To do this, we represent the integral staying in the right-hand side of (5.11) in the form

$$\left(\int_0^{\varepsilon_3}+\int_{\varepsilon_3}^\infty\right)\mathrm{Re}\left(e^{-ix\eta}\frac{\psi_2(\xi+i\eta)}{\psi_2(\xi)}\right)d\eta =: I^{(1)} + I^{(2)}, \qquad (5.12)$$

where $\varepsilon_3 = \varepsilon_3(\xi)$ will be choosen later. Let us estimate $I^{(1)}$ from below and $|I^{(2)}|$ from above to show that $I^{(1)} > 0$ and $I^{(1)} > |I^{(2)}|$ for ξ being large enough.

5.3. Set

$$b_2(\xi) := \log\psi_2(\xi) = \log F(\xi) + h(\xi), \qquad (5.13)$$
$$\kappa(\xi) := \min(\delta(|\xi|),\ h^{-1/8}(|\xi|)). \qquad (5.14)$$

Lemma 6 *The following inequalities hold*

$$b_2(\xi) \ge \frac{2}{3}(\xi^2+1), \quad \xi > 0: \qquad (5.15)$$
$$b_2'(\xi) > 0, \quad \xi > 0; \qquad (5.16)$$
$$b_2''(\xi) > 0, \quad \xi > 0; \qquad (5.17)$$
$$|b_2^{(j)}(\xi)| \le C_{25}j!2^j(\kappa(\xi))^{-j}h(\xi), \quad \xi \ge \xi_1 > 0,\ j = 1,2,\ldots. \qquad (5.18)$$

Proof of Lemma 6. By (5.13) and (5.4),

$$b_2(\xi) = \log F(\xi) + h(\xi) \geq -\frac{1}{3}g(\xi) + h(\xi) \geq \frac{2}{3}h(\xi) \geq \frac{2}{3}(\xi^2+1), \tag{5.19}$$

and (5.15) follows. Using (5.4) and Lemma 5 (a)

$$b_2'(\xi) = (\log F(\xi))' + h'(\xi) \geq -\frac{1}{3(1-\alpha)}h(\xi) + h'(\xi); \tag{5.20}$$

$$b_2''(\xi) = (\log F(\xi))'' + h''(\xi) \geq -h(\xi) + h''(\xi). \tag{5.21}$$

From (5.1) and (4.18) we have

$$h'(\xi) \geq h(\xi)\int_0^1 e^{\xi u}u^{-\alpha}du \geq \frac{1}{1-\alpha}h(\xi), \tag{5.22}$$

$$h''(\xi) \geq h(\xi)\left\{\left(\int_0^1 e^{\xi u}u^{-\alpha}du\right)^2 + \int_0^1 e^{\xi u}u^{1-\alpha}du\right\} \geq h(\xi). \tag{5.23}$$

Substituting (5.22) and (5.23) into (5.20) and (5.21) respectively, we get (5.16) and (5.17).

To prove (5.18) we need

Lemma ([6, Ch.I]) *Let $\nu(z)$, $|\nu(0)| \geq 1$, be an entire function non-vanishing in $\{z : |z| < \rho\}$. Then*

$$\left|\left(\frac{d^j}{dz^j}\log \nu(z)\right)_{z=0}\right| \leq C_{26}j!\rho^{-j}\max_{0\leq\theta\leq 2\pi}\log^+|\nu(\rho e^{i\theta})|, \quad j=1,2,\ldots \tag{5.24}$$

The function $\nu(z) = \psi_2(\xi+z)$ is non-vanishing in $\{z : |z-\xi| < \kappa(\xi)/2\}$ by virtue of (5.14). Moreover, (5.13) and (5.14) imply

$$\nu(0) = \psi_2(\xi) = \exp b_2(\xi) \geq e^{2/3} \geq 1. \tag{5.25}$$

Applying the above lemma, we get

$$|b_2^{(j)}(\xi)| \leq C_{27}j!2^j(\kappa(\xi))^{-j}\max_{0\leq\theta\leq 2\pi}\log^+\left|\psi_2\left(\xi+\frac{\kappa(\xi)}{2}e^{i\theta}\right)\right|, \quad j=1,2,\ldots \tag{5.26}$$

Using (5.13), (5.16) and the evident inequality $|h(x+iy)| \leq h(x)$, we get

$$\log^+\left|\psi_2\left(\xi+\frac{\kappa(\xi)}{2}e^{i\theta}\right)\right| \leq \log^+\left|F\left(\xi+\frac{\kappa(\xi)}{2}e^{i\theta}\right)\right| + \left|h\left(\xi+\frac{\kappa(\xi)}{2}e^{i\theta}\right)\right| \leq$$

$$\frac{4}{3}h\left(\xi+\frac{\kappa(\xi)}{2}\cos\theta\right) \leq \frac{4}{3}h\left(\xi+\frac{\kappa(\xi)}{2}\right), \quad \xi > 0. \tag{5.27}$$

Whence, by (5.14) and Lemma 5 (b),

$$\log^+\left|\psi_2\left(\xi+\frac{\kappa(\xi)}{2}e^{i\theta}\right)\right| \leq \frac{8}{3}h(\xi), \quad \xi \geq \xi_1 > 0, \tag{5.28}$$

Substituting this into (5.26), we get (5.18). □

5.4. The function $\psi_2(z)$ does not vanish in $\{z : |z-\xi| < \kappa(\xi)\}$. Therefore

$$\log\left\{e^{-ix\eta}\frac{\psi_2(\xi+i\eta)}{\psi_2(\xi)}\right\} = -ix\eta + \sum_{j=1}^{\infty}\frac{(i\eta)^j}{j!}b_2^{(j)}(\xi), \quad |i\eta-\xi| < \kappa(\xi). \tag{5.29}$$

From (5.15), (5.16), (5.17) we derive that

$$b_2'(\xi) \uparrow \infty, \quad \text{as} \quad \xi \uparrow +\infty. \tag{5.30}$$

Hence the equation

$$b_2'(\xi) = x \tag{5.31}$$

has a unique solution $\xi = \xi(x)$ for all $x \geq x_0 > 0$. It is evident that

$$\xi(x) \uparrow +\infty, \quad \text{as} \quad x \uparrow +\infty. \tag{5.32}$$

Substituting $\xi = \xi(x)$ into (5.29), we obtain

$$\log\left(e^{-ix\eta}\frac{\psi_2(\xi+i\eta)}{\psi_2(\xi)}\right) = -\frac{b_2''(\xi)}{2}\eta^2 + \tau_2(\xi,\eta), \tag{5.33}$$

$$\tau_2(\xi,\eta) := \sum_{j=3}^{\infty}\frac{(i\eta)^j}{j!}b_2^{(j)}(\xi). \tag{5.34}$$

From (5.18) it follows that

$$|\tau_2(\xi,\eta)| \leq C_{25}h(\xi)\sum_{j=3}^{\infty}\frac{2\eta^j}{\kappa(\xi)^j} = C_{25}h(\xi)\left(\frac{2\eta}{\kappa(\xi)}\right)^3\left(1-\frac{2\eta}{\kappa(\xi)}\right)^{-1}. \tag{5.35}$$

Assuming $0 \leq \eta \leq \kappa(\xi)/4$, we get

$$|\tau_2(\xi,\eta)| \leq C_{28}h(\xi)(\kappa(\xi))^{-3}\eta^3 \tag{5.36}$$

Let us choose

$$\varepsilon_3 = \varepsilon_3(\xi) := \left(\frac{\pi}{3C_{28}}\cdot\frac{1}{h(\xi)}\right)^{1/3}\kappa(\xi). \tag{5.37}$$

Then

$$|\tau_2(\xi,\eta)| \leq \frac{\pi}{3}, \quad \text{for} \quad 0 \leq \eta \leq \varepsilon_3. \tag{5.38}$$

Hence by (5.33)

$$I^{(1)} = \int_0^{\varepsilon_3}\exp\left\{-\frac{b_2''(\xi)}{2}\eta^2 + \mathrm{Re}\tau_2(\xi,\eta)\right\}\cos(\mathrm{Im}\tau_2(\xi,\eta))d\eta \geq \frac{1}{2}e^{-\pi/3}\int_0^{\varepsilon_3}\exp\left(-\frac{b_2''(\xi)}{2}\eta^2\right)d\eta. \tag{5.39}$$

Taking into account (5.17) and (5.18), we obtain

$$I^{(1)} \geq \frac{1}{2}e^{-\pi/3}\int_0^{\varepsilon_3}\exp\left(-C_{29}\eta^2\frac{h(\xi)}{(\kappa(\xi))^2}\right)d\eta = \frac{1}{2}e^{-\pi/3}\frac{\kappa(\xi)}{\sqrt{h(\xi)}}\int_0^{\varepsilon_3\sqrt{h(\xi)}/\kappa(\xi)}\exp(-C_{29}u^2)du \tag{5.40}$$

Evidently, (5.37) implies

$$\varepsilon_3\sqrt{h(\xi)}/\kappa(\xi) = C_{30}(h(\xi))^{1/6} \to \infty, \quad \text{as} \quad \xi \to \infty. \tag{5.41}$$

Hence

$$I^{(1)} \geq C_{31}\kappa(\xi)/\sqrt{h(\xi)}. \tag{5.42}$$

By (5.14) and (5.4), $\kappa(\xi) \geq h^{-1/8}(\xi)$. Thus

$$I^{(1)} \geq C_{31}(h(\xi))^{-5/8}. \tag{5.43}$$

5.5. Evidently, (5.12) and (5.5) imply

$$|I^{(2)}| \leq \int_{\varepsilon_3}^{\infty}\left|\frac{F(\xi+i\eta)}{F(\xi)}\right|\exp\{\mathrm{Re}h(\xi+i\eta)-h(\xi)\}d\eta. \tag{5.44}$$

Lemma 7 *There exists $\xi_2 > 0$ such that, for all $\xi \geq \xi_2,\ \eta \geq \varepsilon_3(\xi)$,*

$$Reh(\xi + i\eta) - h(\xi) \leq -C_{32}h(\xi)\varepsilon_3^2(\xi). \tag{5.45}$$

Proof of Lemma 7. We get

$$\mathrm{Re}h(\xi + i\eta) - h(\xi) =$$

$$\mathrm{Re}\sum_{m=0}^{\infty} a_m \left\{\exp\left(m\int_0^1 (e^{(\xi+i\eta)t} - 1)t^{-1-\alpha}dt\right) - \exp\left(m\int_0^1 (e^{\xi t} - 1)t^{-1-\alpha}dt\right)\right\} \leq$$

$$\sum_{m=0}^{\infty} a_m \exp\left(m\int_0^1 (e^{\xi t} - 1)t^{-1-\alpha}dt\right)\left[\exp\left(\int_0^1 e^{\xi t}(\cos\eta t - 1)t^{-1-\alpha}dt\right) - 1\right].$$

Taking into account that $\xi > 0$ and $\eta \leq \varepsilon_3(\xi)$,

$$\mathrm{Re}h(\xi + i\eta) - h(\xi) \leq -h(\xi)\left[1 - \exp\left(-2\eta^\alpha\int_0^\eta \sin^2\frac{u}{2}\ u^{-1-\alpha}du\right)\right] \leq$$

$$-h(\xi)\left[1 - \exp\left(-2(\varepsilon_3(\xi))^\alpha\int_0^{\varepsilon_3(\xi)} \sin^2\frac{u}{2}\ u^{-1-\alpha}du\right)\right].$$

Since $\varepsilon_3(\xi) < \pi/2$ for ξ being large enough, we get the statement of Lemma 7 by the following way:

$$\mathrm{Re}h(\xi + i\eta) - h(\xi) \leq -h(\xi)\left[1 - \exp\left(-2(\varepsilon_3(\xi))^\alpha\int_0^{\varepsilon_3(\xi)} \frac{1}{\pi^2}u^{1-\alpha}du\right)\right] =$$

$$-h(\xi)\left[1 - \exp(-C_{33}\varepsilon_3^2(\xi))\right] \leq -C_{32}h(\xi)\varepsilon_3^2(\xi). \quad \Box$$

Substituting (5.45) into (5.44) and taking into account (5.4), we obtain

$$|I^{(2)}| \leq \exp(-C_{32}h(\xi)\varepsilon_3^2(\xi))\int_{\varepsilon_3}^{\infty}\left|\frac{F(\xi + i\eta)}{F(\xi)}\right|d\eta \leq h(\xi)\exp(-C_{32}h(\xi)\varepsilon_3^2(\xi)).) \tag{5.46}$$

Noting that (5.34), (5.14), and (5.4) imply

$$C_{32}h(\xi)\varepsilon_3^2(\xi) = C_{34}\sqrt{h(\xi)}(\kappa(\xi))^2 \geq C_{35}(h(\xi))^{1/12}, \tag{5.47}$$

we finally get from (5.46)

$$|I^{(2)}| \leq h(\xi)\exp(-C_{35}(h(\xi))^{5/24}). \tag{5.48}$$

Joining (5.48), (5.43), (5.12) and (5.11), we complete the proof of Theorem 4. □

6 Proof of Lemmas 1 and 5

6.1. Proof of Lemma 1. First we are going to show the validity of (4.8).

By (4.3)

$$\log B_2(-r) = \log C_{36} + \sum_{b_k\in\Lambda_+}\log\frac{|b_k + r|^2}{|b_k + \mu_k + r|^2} = \log C_{36} + \sum_{b_k\in\Lambda_+}\log\left(1 - \frac{\mu_k^2 + 2\mathrm{Re}b_k\mu_k + 2\mu_k r}{|b_k + \mu_k + r|^2}\right) \tag{6.1}$$

From (4.4) we conclude that for all $b_k \in \Lambda_+$

$$\frac{\mu_k^2 + 2\mathrm{Re}b_k\mu_k + 2\mu_k r}{|b_k + \mu_k + r|^2} \leq 2\frac{\mathrm{Re}b_k\mu_k + \mu_k^2}{|b_k|^2} + 2\frac{\mu_k \cdot |b_k|r}{|b_k|(|b_k|^2 + r^2)} \leq \frac{1}{2}. \tag{6.2}$$

Using the inequality

$$\log(1-x) \geq -(2\log 2)x, \quad 0 \leq x \leq 1/2, \tag{6.3}$$

we get

$$\begin{aligned}\log B_2(-r) \quad &\geq \quad \log C_{36} - 2\log 2 \sum_{b_k\in\Lambda_+} \frac{\mu_k^2 + 2\mathrm{Re} b_k \mu_k + 2\mu_k r}{|b_k+\mu_k+r|^2} \geq \\ &\log C_{36} - 4\log 2\left(\sum_{b_k\in\Lambda_+} \frac{\mathrm{Re} b_k\mu_k + \mu_k^2}{|b_k|^2} + r\sum_{b_k\in\Lambda_+}\frac{\mu_k}{|b_k|^2+r^2}\right)\end{aligned} \tag{6.4}$$

Hence by (3.13) and (4.7), we obtain (4.8):

$$\log B_2(-r) \geq -C_2 r\left(\frac{1}{r} + \sum_{b_k\in\Lambda_+}\frac{\mu_k}{|b_k|^2+r^2}\right) \geq -C_2 r Q(r), \quad r\geq 1. \tag{6.5}$$

Let us prove (4.10). From (4.3) it follows

$$\begin{aligned}\left(\frac{d^j}{dz^j}\log B_2(z)\right)_{z=-r} &= (j-1)!\sum_{b_k\in\Lambda_+}\left\{-\frac{1}{(b_k+r)^j} - \frac{1}{(\bar{b}_k+r)^j} + \frac{1}{(b_k+\mu_k+r)^j} + \right. \\ \left.\frac{1}{(\bar{b}_k+\mu_k+r)^j}\right\} &= (j-1)!\sum_{b_k\in\Lambda_+} 2\mathrm{Re}\left(\frac{1}{(b_k+r+\mu_k)^j} - \frac{1}{(b_k+r)^j}\right) = -j!\cdot 2\sum_{b_k\in\Lambda_+}\sigma_k,\end{aligned} \tag{6.6}$$

where

$$\sigma_k := \int_0^{\mu_k}\mathrm{Re}\frac{1}{(b_k+r+u)^{j+1}}du. \tag{6.7}$$

First we estimate σ_k for $j\geq 3$:

$$|\sigma_k| \leq \frac{\mu_k r}{(|b_k|^2+r^2)^2}\cdot\frac{r^{j-3}}{(|b_k|^2+r^2)^{(j-3)/2}}\cdot\frac{1}{r^{j-2}} \leq \frac{\mu_k r}{(|b_k|^2+r^2)^2}\cdot\frac{1}{r^{j-2}}. \tag{6.8}$$

Using (4.6), (6.6), (6.7), we get (4.10) for $j\geq 3$.

Now we consider $j=2$:

$$\begin{aligned}|\sigma_k| \quad &= \quad \left|\int_0^{\mu_k}\frac{(\mathrm{Re} b_k+r+u)((\mathrm{Re} b_k+r+u)^2 - 3(\mathrm{Im} b_k)^2)}{|b_k+r+u|^6}du\right| \leq \\ &3\mu_k\frac{\mathrm{Re} b_k+r+\mu_k}{(|b_k|^2+r^2)^2} \leq \frac{3}{2}\cdot\frac{\mu_k\mathrm{Re} b_k+\mu_k^2}{|b_k|^2 r^2} + 3\frac{\mu_k r}{(|b_k|^2+r^2)^2}.\end{aligned} \tag{6.9}$$

Using (4.6), (3.13), (6.6), (6.7), we get (4.10):

$$\left|(\log B_2(z))''\right|_{z=-r} \leq 2C_4\left(\frac{1}{r^2} + \sum_{b_k\in\Lambda_+}\frac{2\mu_k r}{(|b_k|^2+r^2)^2}\right) \leq 2C_4 q(r), \quad r\geq 1. \tag{6.10}$$

Let us prove (4.9). Substituting $j=1$ into (6.6) and (6.7), we get

$$\begin{aligned}|\sigma_k| \quad &\leq \quad \left|\int_0^{\mu_k}\frac{(\mathrm{Re} b_k+r+u)^2-(\mathrm{Im} b_k)^2}{|b_k+r+u|^4}du\right| \leq \int_0^{\mu_k}\frac{|b_k|^2+2\mathrm{Re} b_k(r+u)+(r+u)^2}{|b_k+r+u|^4}du = \\ &\int_0^{\mu_k}\frac{|b_k+r+u|^2}{|b_k+r+u|^4}du\end{aligned} \tag{6.11}$$

Then we have

$$|\sigma_k| \leq \frac{\mu_k}{|b_k|^2 + r^2}. \tag{6.12}$$

Using (4.7), (6.6), (6.7), we obtain (4.9).

Let us prove (4.11) and (4.12). Substituting $z = -r + iy$, $b_k = \alpha_k + i\beta_k$ ($r > 0$, $\alpha_k > 0$, $\beta_k > 0$, $y > 0$) into (4.3), we get

$$\begin{aligned}
2\log\frac{|B_2(z)|}{B_2(-r)} &= \sum_{b_k\in\Lambda_+}\left(\log\frac{(r+\alpha_k)^2+(y-\beta_k)^2}{(r+\alpha_k+\mu_k)^2+(y-\beta_k)^2} - \log\frac{(r+\alpha_k)^2+\beta_k^2}{(r+\alpha_k+\mu_k)^2+\beta_k^2}\right) + \\
&\sum_{b_k\in\Lambda_+}\left(\log\frac{(r+\alpha_k)^2+(y+\beta_k)^2}{(r+\alpha_k+\mu_k)^2+(y+\beta_k)^2} - \log\frac{(r+\alpha_k)^2+\beta_k^2}{(r+\alpha_k+\mu_k)^2+\beta_k^2}\right) = \\
&\sum_{b_k\in\Lambda_+}\left(\log\frac{(r+\alpha_k)^2+(y-\beta_k)^2}{(r+\alpha_k)^2+\beta_k^2} - \log\frac{(r+\alpha_k+\mu_k)^2+(y-\beta_k)^2}{(r+\alpha_k+\mu_k)^2+\beta_k^2}\right) + \\
&\sum_{b_k\in\Lambda_+}\left(\log\frac{(r+\alpha_k)^2+(y+\beta_k)^2}{(r+\alpha_k)^2+\beta_k^2} - \log\frac{(r+\alpha_k+\mu_k)^2+(y+\beta_k)^2}{(r+\alpha_k+\mu_k)^2+\beta_k^2}\right) = \sum_{b_k\in\Lambda_+}\left(\gamma_k^{(1)}+\gamma_k^{(2)}\right),
\end{aligned} \tag{6.13}$$

where

$$\begin{aligned}
\gamma_k^{(1)} &:= \log\left(1-\frac{2\beta_k y - y^2}{(r+\alpha_k)^2+\beta_k^2}\right) - \log\left(1-\frac{2\beta_k y - y^2}{(r+\alpha_k)^2+\beta_k^2}\cdot\frac{(r+\alpha_k)^2+\beta_k^2}{(r+\alpha_k+\mu_k)^2+\beta_k^2}\right), \\
\gamma_k^{(2)} &:= \log\left(1+\frac{2\beta_k y + y^2}{(r+\alpha_k)^2+\beta_k^2}\right) - \log\left(1+\frac{2\beta_k y + y^2}{(r+\alpha_k)^2+\beta_k^2}\cdot\frac{(r+\alpha_k)^2+\beta_k^2}{(r+\alpha_k+\mu_k)^2+\beta_k^2}\right).
\end{aligned} \tag{6.14}$$

We shall obtain the estimates for $\gamma_k^{(1)}$ and $\gamma_k^{(2)}$ with help of the following elementary inequalities:

$$\log(1-u) - \log(1-au) \leq \frac{u}{au-1}(1-a), \quad \text{for} \quad 0 < u,\ a < 1; \tag{6.15}$$

$$\log(1+u) - \log(1+au) \leq \frac{u}{au+1}(1-a), \quad \text{for} \quad u > 0,\ 0 < a < 1. \tag{6.16}$$

Let us consider $\gamma_k^{(1)}$. If $\beta_k > y/2$, then

$$0 < \frac{2\beta_k y - y^2}{(r+\alpha_k)^2+\beta_k^2} = \frac{\beta_k^2 - (y-\beta_k)^2}{(r+\alpha_k)^2+\beta_k^2} \leq \frac{\beta_k^2}{(r+\alpha_k)^2+\beta_k^2} < 1, \tag{6.17}$$

and we use (6.15) with

$$u = \frac{|y^2 - 2\beta_k y|}{(r+\alpha_k)^2+\beta_k^2}, \quad a = \frac{(r+\alpha_k)^2+\beta_k^2}{(r+\alpha_k+\mu_k)^2+\beta_k^2}. \tag{6.18}$$

If $\beta_k \leq y/2$, then $y^2 - 2\beta_k y \geq 0$, and we use (6.16) with u and a defined by (6.18). In both cases we obtain

$$\gamma_k^{(1)} \leq \frac{2\mu_k r + 2\mu_k\alpha_k + \mu_k^2}{(r+\alpha_k)^2+\beta_k^2}\cdot\frac{y^2 - 2\beta_k y}{(r+\alpha_k+\mu_k)^2+(y-\beta_k)^2}. \tag{6.19}$$

To estimate $\gamma_k^{(2)}$, we use (6.16) with

$$u = \frac{2\beta_k y + y^2}{(r+\alpha_k)^2+\beta_k^2}, \quad a = \frac{(r+\alpha_k)^2+\beta_k^2}{(r+\alpha_k+\mu_k)^2+\beta_k^2}. \tag{6.20}$$

We obtain

$$\gamma_k^{(2)} \leq \frac{2\mu_k r + 2\mu_k\alpha_k + \mu_k^2}{(r+\alpha_k)^2+\beta_k^2}\cdot\frac{y^2 + 2\beta_k y}{(r+\alpha_k+\mu_k)^2+(y+\beta_k)^2}. \tag{6.21}$$

Joining (6.21), (6.19) and (6.13), we get

$$\log \frac{|B_2(z)|}{B_2(-r)} \le 2y^2 \sum_{b_k\in\Lambda_+} \frac{2\mu_k r + 2\mu_k\alpha_k + \mu_k^2}{(r+\alpha_k)^2+\beta_k^2} D_k, \tag{6.22}$$

$$D_k := \frac{(r+\alpha_k+\mu_k)^2 + y^2 - 3\beta_k^2}{[(r+\alpha_k+\mu_k)^2+(y-\beta_k)^2][(r+\alpha_k+\mu_k)^2+(y+\beta_k)^2]}. \tag{6.23}$$

Set

$$M := \{k : b_k \in \Lambda_+,\ (r+\alpha_k+\mu_k)^2 + y^2 - 3\beta_k^2 > 0\}, \tag{6.24}$$

Then (6.22) can be rewritten in the form

$$\log \frac{|B_2(z)|}{B_2(-r)} \le y^2 \sum_{k\in M} \frac{2\mu_k r + 2\mu_k\alpha_k + \mu_k^2}{r^2+|b_k|^2} D_k. \tag{6.25}$$

Lemma 8 . *For each $k \in M$ the following estimations hold*

$$D_k \le \frac{1}{r^2+|b_k|^2}, \quad \textit{for} \quad r>0,\ y>0; \tag{6.26}$$

$$D_k \le \frac{Cy^2}{(y^2+|b_k|^2)^2}, \quad \textit{for} \quad y>0,\ 1\le r\le \frac{y}{2}. \tag{6.27}$$

Proof. Evidently,

$$(r^2+|b_k|^2)D_k = \frac{(r^2+\alpha_k^2+\beta_k^2)[(r+\alpha_k+\mu_k)^2+y^2-3\beta_k^2]}{[(r+\alpha_k+\mu_k)^2+(y-\beta_k)^2][(r+\alpha_k+\mu_k)^2+(y+\beta_k)^2]}. \tag{6.28}$$

Denoting $x_k := r+\alpha_k+\mu_k$, we get

$$(r^2+|b_k|^2)D_k \le \frac{(x_k^2+\beta_k^2)(x_k^2+y^2-3\beta_k^2)}{[x_k^2+(y-\beta_k)^2][x_k^2+(y+\beta_k)^2]}. \tag{6.29}$$

To get (6.26), it suffices to show that

$$\frac{(x_k^2+\beta_k^2)(x_k^2+y^2-3\beta_k^2)}{[x_k^2+(y-\beta_k)^2][x_k^2+(y+\beta_k)^2]} \le 1. \tag{6.30}$$

This inequality is equivalent to

$$x_k^2(y^2+4\beta_k^2) + (y^2-\beta_k^2)^2 - \beta_k^2(y^2-\beta_k^2) + 2\beta_k^4 \ge 0. \tag{6.31}$$

The last inequality follows from the calculation:

$$x_k^2(y^2+4\beta_k^2) + (y^2-\beta_k^2)^2 - \beta_k^2(y^2-\beta_k^2) + 2\beta_k^4 \ge (y^2-\beta_k^2)^2 - \beta_k^2(y^2-\beta_k^2) + 2\beta_k^4 =$$
$$\beta_k^4\left(\left(\frac{y^2-\beta_k^2}{\beta_k^2}\right)^2 - \frac{y^2-\beta_k^2}{\beta_k^2} + 2\right) > 0 \quad \text{for} \quad x_k>0,\ y>0,\ \beta_k>0. \tag{6.32}$$

This completes the proof of (6.26).

Let us prove (6.27). For $r \le y/2,\ \beta_k \ge \sqrt{2/3}y$, (4.4) implies

$$(r+\alpha_k+\mu_k)^2 + y^2 - 3\beta_k^2 \le 3r^2 + 3\alpha_k^2 + 3\mu_k + y^2 - 3\beta_k^2 \le \frac{7}{4}y^2 - \frac{21}{8}\beta_k^2 \le 0. \tag{6.33}$$

Hence $M \subset \{k : \beta_k \le \sqrt{2/3}y\}$. Therefore, for each $k \in M$,

$$\frac{(y-\beta_k)^2}{y^2+\beta_k^2} = 1 - \frac{2(y/\beta_k)}{(y/\beta_k)^2+1} \ge 1 - \frac{2\sqrt{3/2}}{5/2} =: C_{37} > 0. \tag{6.34}$$

Hence

$$\begin{aligned} \left[(r+\alpha_k+\mu_k)^2+(y+\beta_k)^2\right]\left[(r+\alpha_k+\mu_k)^2+(y-\beta_k)^2\right] &\ge \\ \left[y^2+\alpha_k^2+\beta_k^2\right]\left[\alpha_k^2+(y-\beta_k)^2\right] &\ge C_{37}(y^2+\beta_k^2)^2. \end{aligned} \tag{6.35}$$

Using (6.33) and (6.23), we get

$$D_k \le \frac{(7/4)y^2-(21/8)\beta_k^2}{C_{37}(y^2+\beta_k^2)^2} \le \frac{(7/4)y^2}{C_{37}(y^2+\beta_k^2)^2}. \tag{6.36}$$

This completes the proof of (6.27) and Lemma 8. □

6.2. To complete proof of (4.11) we substitute (6.26) into (6.25) and use (3.13) and (4.6). Then we get (4.11) from the following calculation:

$$\begin{aligned} \log\frac{|B_2(z)|}{B_2(-r)} &\le y^2\left(\sum_{k\in M}\frac{2\mu_k r}{(r^2+|b_k|^2)^2}+\sum_{k\in M}\frac{2\mu_k \mathrm{Re} b_k+\mu_k^2}{(r^2+|b_k|^2)^2}\right) \le \\ & C_{38}y^2\left(\sum_{b_k\in\Lambda_+}\frac{2\mu_k r}{(r^2+|b_k|^2)^2}+\frac{1}{r^2}\right) \le C_{38}y^2 q(r). \end{aligned} \tag{6.37}$$

To complete proof of (4.12), we substitute (6.27) into (6.25) and use (3.13) and (4.6). We get (4.12) from the following calculation:

$$\begin{aligned} \log\frac{|B_2(z)|}{B_2(-r)} &\le C_{39}y^2\left(\sum_{k\in M}\frac{\mu_k^2+2\mathrm{Re} b_k\mu_k}{r^2+|b_k|^2}\cdot\frac{y^2}{(y^2+|b_k|^2)^2}+\sum_{k\in M}\frac{2\mu_k r}{r^2+|b_k|^2}\cdot\frac{y^2}{(y^2+|b_k|^2)^2}\right) \le \\ & C_{40}y^2\left(\frac{1}{y^2}+\sum_{b_k\in\Lambda_+}\frac{\mu_k}{|b_k|}\cdot\frac{1}{y^2+|b_k|^2}\right) \le C_{40}y^2 q(y). \end{aligned} \tag{6.38}$$

6.3. Proof of Lemma 5. We need the following result.

Lemma [3]. *Let $s(r) \ge 1$ be a continuous nondecreasing function on R^+. For any given $\delta > 0$ and $K_0 > 1$, there exists an entire function $y(z)$ with nonnegative Taylor coefficients such that*

$$y(r) \ge s(r), \quad \text{for} \quad r \ge 0; \tag{6.39}$$

$$y(r + y^{-\delta}(r)) \le K_0 y(r), \quad \text{for} \quad r \ge r_0. \tag{6.40}$$

Let $y(z)$ be an entire function of Lemma with

$$s(r) = \max\{g(r), [\varphi_\alpha(r)e^{r+2}/(1-\alpha)]^{2/\beta}\},\ \delta = \frac{\beta}{2},\ K_0 = 2. \tag{6.41}$$

Set

$$h(z) = y(\varphi_\alpha(z)). \tag{6.42}$$

To check Lemma 5 for this h one needs just elementary estimations.

7 Proof of Theorem 5

7.1. We shall reduce the proof to a proposition related to positivity of a system of polynomials on R^+.
First we shall deduce Theorem 5 from the following proposition.

Proposition 1 *Let $p : [a, b] \to R$ be a continuous function satisfying the following conditions:*

(a)

$$\psi(u) := \int_a^b e^{ux} p(x)dx > 0, \quad \forall u \in R; \tag{7.1}$$

(b)

$$p(a) > 0, \ p(b) > 0. \tag{7.2}$$

Then $(\exists \varepsilon_0 > 0)$ $(\forall \varepsilon \in (0, \varepsilon_0])$ $(\exists \lambda_0 > 0)$ $(\forall \lambda \geq \lambda_0)$ *the function*

$$\psi_{\lambda,\varepsilon}(z) := \psi(z) \exp(\lambda e^{\varepsilon z}) \tag{7.3}$$

admits the representation by the absolutely convergent integral

$$\psi_{\lambda,\varepsilon}(z) = \int_0^\infty e^{zx} w_{\lambda,\varepsilon}(x)dx, \tag{7.4}$$

where $w_{\lambda,\varepsilon}(x) \geq 0$ *for* $x \geq 0$.

7.2. Set

$$\psi_R(u) := \int_{1/R}^R e^{ux} p(x)dx, \quad R > 0, \tag{7.5}$$

where $p(x)$ is a function satisfying the conditions of Theorem 5. To deduce Theorem 5 from Proposition 1, we need the following lemma:

Lemma 9 .
There exists $R_0 > \max(x_2, 1/x_1)$ such that $\psi_R(u) > 0$ for $R \geq R_0$ and $u \in R$.

Proof of Lemma 9. First we show that there exist $M > 0$ and $R_1 > 0$ such that $\psi_R(u) > 0$ for $|u| \geq M, \ R \geq R_1$.

Let $\rho > 0$ be a number such that $p(u) > 0$ for $u \in (0, 1/\rho) \bigcup (\rho, \infty)$. Fix $R_1 > \rho$. Evidently, for all $R \geq R_1$

$$\psi_R(u) \geq \psi_{R_1}(u) = \int_{1/R_1}^{R_1} e^{ux} p(x)dx, \quad u \in R. \tag{7.6}$$

For sufficiently large $u > 0$

$$\psi_{R_1}(u) \geq \int_{(\rho+R_1)/2}^{R_1} e^{ux} p(x)dx - \int_{1/R_1}^{\rho} e^{ux}|p(x)|dx \geq C_{41} e^{(\rho+R_1)u/2} - C_{42} e^{\rho u} > 0. \tag{7.7}$$

For sufficiently large $|u|, \ u < 0$,

$$\psi_{R_1}(u) \geq \int_{1/R_1}^{[(1/R_1)+(1/\rho)]/2} e^{ux} p(x)dx - \int_{1/\rho}^{R_1} e^{ux}|p(x)|dx > 0 \tag{7.8}$$

From (7.6), (7.7) and (7.8) we conclude that there exists a constant $M > 0$ not depending on R such that

$$\psi_R(u) > 0, \quad \text{for} \quad |u| \geq M, \ R \geq R_1. \tag{7.9}$$

It remains to prove that there exists $R_0 \geq R_1$ such that, for all $R \geq R_0$, $|u| \leq M$, we have $\psi_R(u) > 0$. From (a) of Theorem it follows that

$$\lim_{R\to\infty} \psi_R(u) = \psi(u) = \int_0^\infty e^{ux} p(x)dx, \tag{7.10}$$

where the limit is uniform on any compact subset of R. From (b) of Theorem 5 and (7.10) it follows that there exists $R_2 > 0$ such that $\psi_R(u) > 0$ for $R \geq R_2$, $|u| \leq M$. Hence $\psi_R(u) > 0$ for all $u \in R$, $R \geq R_0 := \max(R_1, R_2)$. □

7.3. Let us deduce Theorem 5 from Proposition 1 with help of Lemma 9. Let

$$p(x) =: p_1(x) + p_2(x) + p_3(x), \tag{7.11}$$

where

$$p_1(x) = 0 \quad \text{for} \quad x \in (\frac{1}{R_0}, \infty); \tag{7.12}$$

$$p_2(x) = 0 \quad \text{for} \quad x \in (0, \frac{1}{R_0}) \bigcup (R_0, \infty); \tag{7.13}$$

$$p_3(x) = 0 \quad \text{for} \quad x \in (0, R_0). \tag{7.14}$$

Evidently,

$$p_1(x) \geq 0, \quad p_3(x) \geq 0, \quad \text{for} \quad x \geq 0. \tag{7.15}$$

Let us write

$$\exp(\lambda e^{\varepsilon x}) = \sum_{k=0}^{\infty} \frac{\lambda^k e^{\varepsilon k x}}{k!} = \int_0^\infty e^{xu} d\Phi_{\lambda,\varepsilon}(u), \tag{7.16}$$

where $\Phi_{\lambda,\varepsilon}(u)$ is a nondecreasing jump function. Then the function

$$\psi_{\lambda,\varepsilon}(u) = \psi(u) \exp(\lambda e^{\varepsilon u}) \tag{7.17}$$

admits the representation

$$\psi_{\lambda,\varepsilon}(u) = \int_0^\infty e^{ux} w_{\lambda,\varepsilon}(x)dx, \tag{7.18}$$

where

$$w_{\lambda,\varepsilon}(x) = (p * \Phi_{\lambda,\varepsilon})(x) = (p_1 * \Phi_{\lambda,\varepsilon})(x) + (p_2 * \Phi_{\lambda,\varepsilon})(x) + (p_3 * \Phi_{\lambda,\varepsilon})(x). \tag{7.19}$$

Evidently, (7.16) implies

$$(p_1 * \Phi_{\lambda,\varepsilon})(x) \geq 0, \quad (p_3 * \Phi_{\lambda,\varepsilon})(x) \geq 0, \quad \forall x \in R. \tag{7.20}$$

To prove Theorem 5, it suffices to show that

$$(\exists \varepsilon_0 > 0)(\forall \varepsilon \in (0, \varepsilon_0])(\exists \lambda_0 > 0)(\forall \lambda \geq \lambda_0)(\forall x \in R)((p_2 * \Phi_{\lambda,\varepsilon})(x)) \geq 0). \tag{7.21}$$

Lemma 9 shows that $p_2(x)$ satisfies conditions of Proposition 1. Therefore (7.21) is an immediate corollary of Proposition 1. Thus, the deduction of Theorem 5 from Proposition 1 is completed. □

7.4. Let us start with the proof of Proposition 1. Without loss of generality, we assume that $[a, b] = [0, 1]$. The condition (b) of Proposition 1 and the continuity of $p(x)$ imply that there are ζ, γ $0 < \zeta < 1/4$, $3/4 < \gamma < 1$, such that

$$p(t) > 0, \quad \text{for} \quad t \in [0, \zeta] \bigcup [\gamma, 1]. \tag{7.22}$$

We have

$$\psi_{\lambda,\varepsilon}(u) = \psi(u) \exp(\lambda e^{\varepsilon u}) = \int_0^\infty e^{ux} w_{\lambda,\varepsilon}(x)dx, \tag{7.23}$$

where

$$w_{\lambda,\varepsilon}(x) = (p * \Phi_{\lambda,\varepsilon})(x) \tag{7.24}$$

(we set $p(t) = 0$ for $t \in (-\infty, 0) \bigcup (1, \infty)$). Further

$$w_{\lambda,\varepsilon}(x) = \int_0^\infty p(x-t) d\Phi_{\lambda,\varepsilon}(t) = \int_{(x-1)^+}^x p(x-t) d\Phi_{\lambda,\varepsilon}(t), \tag{7.25}$$

where as usually $a^+ := \max(a, 0)$. Since $\Phi_{\lambda,\varepsilon}(t) = 0$ for $t < 0$,

$$w_{\lambda,\varepsilon}(x) = 0, \quad \text{for} \quad x \leq 0; \tag{7.26}$$

$$w_{\lambda,\varepsilon}(x) > 0, \quad \text{for} \quad x \in (0, \zeta]. \tag{7.27}$$

We need some additional notations:

(a) $m := \max\{k : k \in Z, x - k\varepsilon \geq 0\}$;

(b) $\delta := x - m\varepsilon$

(c) $n := \{$ the number of integers k such that $k\varepsilon \in [(x-1)^+, x]\}$.

Remembering the definition (7.16) of $\Phi_{\lambda,\varepsilon}$, we can rewrite (7.25) in the following form:

$$\begin{aligned} w_{\lambda,\varepsilon}(x) = p(\delta)\frac{\lambda^m}{m!} + p(\delta+\varepsilon)\frac{\lambda^{m-1}}{(m-1)!} + \ldots + \\ p(\delta + (n-1)\varepsilon)\frac{\lambda^{m-n+1}}{(m-n+1)!} = \frac{\lambda^m}{m!}\Bigg[p(\delta) + p(\delta+\varepsilon)\frac{m}{\lambda} + \\ p(\delta+2\varepsilon)\frac{m}{\lambda}\left(\frac{m}{\lambda} - \frac{1}{\lambda}\right) + \ldots + p(\delta+(n-1)\varepsilon)\frac{m}{\lambda}\left(\frac{m}{\lambda} - \frac{1}{\lambda}\right)\cdots\left(\frac{m}{\lambda} - \frac{n-2}{\lambda}\right)\Bigg]. \end{aligned} \tag{7.28}$$

Let us introduce the following polynomials in y:

$$Q_n(y, \mu, \beta) = p(\mu) + p(\mu+\varepsilon)y + p(\mu+2\varepsilon)y(y-\beta) + \ldots + p(\mu+(n-1)\varepsilon)y(y-\beta)\cdots(y-(n-2)\beta), \tag{7.29}$$

Taking into account the definition of $n(\varepsilon, x)$, we can rewrite (7.28) in the following form:

$$w_{\lambda,\varepsilon}(x) = \frac{\lambda^m}{m!} Q_n\left(\frac{m}{\lambda}, \delta, \frac{1}{\lambda}\right). \tag{7.30}$$

That is why Proposition 1 follows from

Proposition 2 *For $n = n(\varepsilon, \mu) = [(1-\mu)/\varepsilon] + 1$*

$$(\exists \varepsilon_0 \in (0,1))(\forall \varepsilon \in (0, \varepsilon_0])(\exists \beta_0 > 0)(\forall \beta \in [0, \beta_0])(\forall \mu \in [0, \varepsilon))(\forall y \geq 0)(Q_n(y, \mu, \beta) > 0). \tag{7.31}$$

7.5. Set

$$\tilde{Q}_n(y, \mu) := p(\mu) + p(\mu+\varepsilon)y + p(\mu+2\varepsilon)y^2 + \ldots + p(\mu+(n-1)\varepsilon)y^{n-1}. \tag{7.32}$$

Proposition 3 *There exist positive $\bar{\varepsilon}_0, \nu_0$ such that for all $\varepsilon \in (0, \bar{\varepsilon}_0]$, $\mu \in [0, \varepsilon]$, $y \geq 0$, the following inequality holds*

$$\tilde{Q}_n(y, \mu) \geq \nu_0 > 0, \tag{7.33}$$

where $n = n(\varepsilon, \mu) = [(1-\mu)/\varepsilon] + 1$.

Let us deduce Proposition 2 from Proposition 3. Let $\varepsilon_0 := \min\{\bar{\varepsilon}_0, (1-\gamma)/4)\}$, where γ is taken from (7.22). We fix arbitrary $\varepsilon \in (0, \varepsilon_0]$. Evidently, degrees of polynomials (7.29) are uniformly bounded (by $1 + 1/\varepsilon$). Let us show that, for fixed $\varepsilon \in (0, \varepsilon_0]$, positive zeros of the polynomials $\tilde{Q}_n$ have a common upper bound not depending on $\mu \in [0, \varepsilon]$ and $\beta \in [0, 1]$. We use the following (well-known)

Lemma. *Let*

$$P(z) = a_0 + a_1 z + \ldots + a_n z^n, \quad a_n \neq 0, \tag{7.34}$$

be a polynomial. Then

$$|P(z)| > 0, \quad \textit{for} \quad |z| > M := 1 + \frac{1}{|a_n|} \max\{|a_j| : 1 \leq j \leq n-1\}. \tag{7.35}$$

The senior coefficient of $Q_n(y, \mu, \beta)$ equals $p(\mu + (n-1)\varepsilon)$, and $\mu + (n-1)\varepsilon \in [\gamma, 1]$.

$$p(\mu + (n-1)\varepsilon) \geq \min\{p(t) : \ t \in [\gamma, 1]\} > 0. \tag{7.36}$$

The coefficients of $Q_n(y, \mu, \beta)$ are bounded from above by a constant depending only on ε. By the last Lemma,

$$Q_n(y, \mu, \beta) > 0, \quad \text{for} \quad y \geq M_0(\varepsilon), \ 0 \leq \mu \leq \varepsilon, \ 0 \leq \beta \leq 1. \tag{7.37}$$

By (7.29) and $n \leq 1 + 1/\varepsilon$,

$$\lim_{\beta \to 0} Q_n(y, \mu, \beta) = \tilde{Q}_n(y, \mu), \tag{7.38}$$

and the limit is uniform with respect to $(y, \mu) \in K \times [0, 1]$, where K is any compact subset of C. Hence by (7.33) we conclude that there exists $\beta_0(\varepsilon) \in (0, 1)$ such that

$$Q_n(y, \mu, \beta) \geq \frac{\nu_0}{2}, \quad \text{for} \quad 0 \leq \beta \leq \beta_0, \ 0 \leq \mu \leq 1, \ 0 \leq y \leq M_0(\varepsilon). \tag{7.39}$$

The deduction of Proposition 2 from Proposition 3 is completed.

7.6. It remains to prove Proposition 3. To this end we need the following Lemma.

Lemma 10 *Let*

$$F(t) = a_0 + a_1 z + \ldots + a_{n-1} z^{n-1} \tag{7.40}$$

be a polynomial with real coefficients. Assume there is $q \in \mathbf{N}, \ q < n, \ n < 2q$ *such that*

$$a_q > 0, \ a_{q+1} > 0, \ \ldots, \ a_{n-1} > 0. \tag{7.41}$$

Assume, moreover,

$$\max\{|a_j| : 0 \leq j \leq q-1\} \leq B, \quad \min\{a_j : q \leq j \leq n-1\} \geq b. \tag{7.42}$$

Then

$$F(t) > 0 \quad \textit{for} \quad t \geq \max\left\{1, \left(\frac{Bq}{b(n-q)}\right)^{1/(n-q)}\right\}. \tag{7.43}$$

Proof. We have

$$F(t) \geq b(t^{n-1} + t^{n-2} + \ldots + t^q) - B(t^{q-1} + t^{q-2} + \ldots + 1) = b\frac{t^{n-q} - 1}{t-1}\left(t^q - B\frac{t^q - 1}{t^{n-q} - 1}\right). \tag{7.44}$$

For q satisfying the conditions of Lemma 10,

$$\frac{t^q - 1}{t^{n-q} - 1} \leq \frac{q}{n-q} t^{2q-n}, \quad \text{for} \quad t \geq 1. \tag{7.45}$$

Hence

$$F(t) \geq b\frac{t^{n-q}-1}{t-1}\left(t^q - \frac{Bq}{b(n-q)}t^{2q-n}\right) > 0 \tag{7.46}$$

under condition (7.43).

Proof of Proposition 3. Let

$$\begin{aligned} B &:= \max\{|p(t)| : 0 \leq t \leq 1\}; & (7.47)\\ b &:= \min\{p(t) : t \in [0,\zeta]\bigcup[\gamma,1]\}; & (7.48)\\ \varepsilon^{(1)} &:= \min\left(\frac{\zeta}{4}, \frac{1-\gamma}{4}\right). & (7.49)\end{aligned}$$

We are going to apply Lemma 10 to

$$F_1(t) = \tilde{Q}_n(y,\mu) - \frac{b}{2}, \quad F_2(t) = y^{n-1}\left(\tilde{Q}_n(\frac{1}{y},\mu) - \frac{b}{2}\right). \tag{7.50}$$

with $\varepsilon \in (0, \varepsilon^{(1)})$.

Let us estimate n and q for $F_1(t)$ in Lemma 10. Note that n is the total number of coefficients of $F_1(t)$ and $n-q$ is the number of positive senior ones, q is the number of other ones. The coefficients of $\tilde{Q}_n(y,\mu)$ have the form $p(\mu+k\varepsilon)$. Therefore (7.22) shows that senior coefficients of $\tilde{Q}_n(y,\mu)$ are positive if k is such that $\gamma \leq \mu + k\varepsilon \leq 1$. Hence, $n-q \geq (1-\gamma)/\varepsilon - 1$. Analogously, $q \leq \gamma/\varepsilon + 1$ and $n \leq 1/\varepsilon + 1$

Applying Lemma 10 to $F_1(y)$, we get

$$F_1(y) > 0 \quad \text{for} \quad y \geq \max\left\{1, \left(\frac{2B(\gamma/\varepsilon+1)}{b((1-\gamma)/\varepsilon-1)}\right)^{((1-\gamma)/\varepsilon-1)^{-1}}\right\}. \tag{7.51}$$

Hence there exists a positive constant C_{45} not depending on ε and μ such that

$$\tilde{Q}_n(y,\mu) > \frac{b}{2}, \quad \text{for} \quad y \geq \exp(C_{45}\varepsilon). \tag{7.52}$$

Applying Lemma 10 to $F_2(y)$, we get analogously that there exists a positive constant C_{46} not depending on ε and such that

$$\tilde{Q}_n(y,\mu) > \frac{b}{2}, \quad \text{for} \quad 0 \leq y \leq \exp(-C_{46}\varepsilon). \tag{7.53}$$

It remains to prove the boundedness of $\tilde{Q}_n(y,\mu)$ from below for $\exp(-C_{46}\varepsilon) \leq y \leq \exp(C_{45}\varepsilon)$. Set

$$q_n(u,\mu) := \tilde{Q}_n(e^{u\varepsilon},\mu) = p(\mu) + p(\mu+\varepsilon)e^{u\varepsilon} + p(\mu+2\varepsilon)e^{2u\varepsilon} + \ldots + p(\mu+(n-1)\varepsilon)e^{(n-1)u\varepsilon}. \tag{7.54}$$

By (7.52) and (7.53),

$$q_n(u,\mu) > \frac{b}{2}, \quad \text{for} \quad |u| > C_{47} := \max(C_{45}, C_{46}), \tag{7.55}$$

where constant C_{47} does not depend on ε and μ.

Let us consider $q_n(u,\mu)$ for $|u| < C_{47}$. From Proposition 1 (a) it follows that there exists positive constant C_{48} such that

$$\psi(u) = \int_0^1 e^{ux}p(x)dx \geq C_{48}, \quad \text{for} \quad |u| \leq C_{47}. \tag{7.56}$$

Note that

$$\begin{aligned}\varepsilon\exp(u\mu)q_n(u,\mu) &= \varepsilon(p(\mu)e^{u\mu} + p(\mu+\varepsilon)e^{u(\mu+\varepsilon)} + p(\mu+2\varepsilon)e^{u(\mu+2\varepsilon)} + \ldots + \\ &\quad p(\mu+(n-1)\varepsilon)e^{u(\mu+(n-1)\varepsilon)}) \to \int_0^1 e^{ux}p(x)dx, \quad \text{as} \quad \varepsilon \to 0 \qquad (7.57)\end{aligned}$$

uniformly with respect to $u \in [-C_{47}, C_{47}]$. We conclude that there exists $\varepsilon^{(2)} > 0$ such that for all $\varepsilon \in (0, \varepsilon^{(2)})$

$$\varepsilon \exp(u\mu) q_n(u,\mu) \geq C_{48}/2, \quad \text{for} \quad |u| \leq C_{47},\ \mu \in [0,\varepsilon]. \tag{7.58}$$

Thus

$$q_n(u,\mu) \geq C_{48} \frac{\exp(-C_{47}\varepsilon^{(2)})}{2\varepsilon^{(2)}}, \quad \text{for} \quad |u| < C_{47}. \tag{7.59}$$

Setting

$$\bar{\varepsilon}_0 = \min(\varepsilon^{(1)}, \varepsilon^{(2)}), \quad \nu_0 = \min\left(\frac{b}{2}, \frac{\exp(-C_{47}\varepsilon^{(2)})}{2\varepsilon^{(2)}}\right), \tag{7.60}$$

we deduce Proposition 3 from (7.54), (7.55) and (7.59). □

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