

A Continual Analogue of a Theorem by M. Fekete and G. Pólya

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1 Introduction and statement of results

In 1912 M. Fekete and G. Pólya [1] proved the following theorem.

Theorem A *Let $f(u) = p_0 + p_1u + \dots + p_nu^n$ be a polynomial of degree n with real coefficients such that*

$$f(u) > 0 \quad \text{for any } u \geq 0. \quad (1)$$

Then there exists a positive number λ_ε such that for any $\lambda \geq \lambda_\varepsilon$ the entire function $\exp(\lambda u)f(u)$ has nonnegative Taylor coefficients.

We are going to give a continual analogue of this theorem. For a function $f : [0; +\infty) \rightarrow R$ we denote by $L(x, f)$ its Laplace transform : $L(x, f) := \int_0^\infty \exp(xt)f(t)dt$.

The main result of this paper is the following

Theorem 1 *Let $p : [0; +\infty) \rightarrow R$ be a function satisfying the following conditions:*

- (a) $p \in C[0; +\infty)$,
- (b) $L(x, |p|) < \infty$ for any $x \geq 0$,
- (c) there exist real numbers a, b , $0 < a \leq b < +\infty$, such that $p(t) \geq 0$ for $t \in [0, a] \cup [b, +\infty)$,

(d) there exist $t_1 \in [0, a]$ and $t_2 \in [b, +\infty)$, such that $p(t_1) > 0$, $p(t_2) > 0$,

(e) $L(x, p) > 0$ for any $x \in R$.

Then $(\exists \varepsilon_0 > 0) (\forall \varepsilon \in (0; \varepsilon_0]) (\exists \lambda_\varepsilon > 0) (\forall \lambda \geq \lambda_\varepsilon)$ the function

$$F_{\lambda, \varepsilon}(x) = L(x, p) \exp(\lambda e^{\varepsilon x}) \quad (2)$$

is the Laplace transform of a nonnegative function $q_{\lambda, \varepsilon}(t)$.

The following theorem about compact supported functions is an immediate corollary of Theorem 1.

Theorem 2 Let $p : [a; b] \rightarrow R$ be a function satisfying the following conditions:

(a) $p \in C[a; b]$,

(b) $p(a) > 0$, $p(b) > 0$,

(c) $L(x, p) > 0$ for any $x \in R$.

Then $(\exists \varepsilon_0 > 0) (\forall \varepsilon \in (0; \varepsilon_0]) (\exists \lambda_\varepsilon > 0) (\forall \lambda \geq \lambda_\varepsilon)$ the function

$$F_{\lambda, \varepsilon}(x) = L(x, p) \exp(\lambda e^{\varepsilon x}) \quad (3)$$

is the Laplace transform of a nonnegative function $q_{\lambda, \varepsilon}(t)$.

This theorem can be viewed as a continual analogue of Theorem A by M. Fekete and G. Pólya. In fact, if we interpret coefficients of the polynomial f as values of the discrete measure μ , supported by the set $\{0, 1, 2, \dots, n\}$, $\mu(\{k\}) = p_k$, then the Laplace transform of this measure at point $x = \log u$, $u > 0$, will be a polynomial $f(u)$. In this interpretation condition (c) of Theorem 2 transforms into condition (1) of Theorem A. Note that (1) implies $p_0 > 0$ and $p_n > 0$ which is an analogue of condition (b) of Theorem 2.

Condition (a) in Theorems 1, 2 could be weakened, but then the proof would be too cumbersome.

The function $\exp(\lambda e^{\varepsilon x})$ is the Laplace transform of the measure $\mu_{\lambda, \varepsilon}$ supported by $\{\varepsilon k\}_{k=0}^{\infty}$, $\mu_{\lambda, \varepsilon}(\{\varepsilon k\}) = \lambda^k / k!$. Therefore Theorems 1 and 2 mean that the convolution $p * \mu_{\lambda, \varepsilon}$ is nonnegative on $[0, \infty)$ for sufficiently small ε and large λ . Conditions on a function p providing the existence of a non-negative Borel measure μ such that $p * \mu$ is nonnegative were investigated by H. Diamond and M. Essén [2]. One of distinctions from their results is that the measure $\mu_{\lambda, \varepsilon}$ in Theorems 1 and 2 is "standard" i.e. its form does not depend on p . More essential distinction is that the Laplace transform of $\mu_{\lambda, \varepsilon}$ does not vanish in the whole plane and hence the zero-sets of $L(x, p)$ and $L(x, p * \mu_{\lambda, \varepsilon})$ coincide. In a bit different situation, when p and μ are supported by the whole real line R , such a preservation of zero-sets played an important role in the solution (given by I.P. Kamynin and I.V. Ostrovskii [4]) to the problem of characterization of

zero-sets of entire Fourier transforms of nonnegative measures supported by R . Therefore Theorems 1 and 2 are of interest in connection with the open problem [3] of characterization of zero-sets of entire Fourier transforms of nonnegative measures supported by a half-line or a finite interval.

The rest of this paper is devoted to the proof of Theorems 1 and 2.

2 Equivalence of Theorems 1 and 2

It is obvious that Theorem 2 is a particular case of Theorem 1. Let us show that Theorem 1 follows from Theorem 2. We shall suppose that the statement of Theorem 2 is true and the conditions of Theorem 1 are satisfied. Denote by l and r the left and right endpoints of the interval supporting the function p , $0 \leq l \leq t_1$, $t_2 \leq r \leq \infty$ (the numbers t_1 and t_2 are from condition (d) of Theorem 1). It is clear that

$$f(x) := \int_l^r \exp(xt)p(t)dt > 0, x \in R. \quad (4)$$

Since the number r is the right boundary point supporting p (maybe $r = \infty$) and the conditions (c) and (d) of Theorem 1 are satisfied, it follows that there exists a sequence of segments $\{[r_k^i, r_k^{ii}]\}_{k=1}^\infty$, $r_1^i < r_1^{ii} < r_2^i < r_2^{ii} < \dots$, $\lim_{k \rightarrow +\infty} r_k^{ii} = r$, such that $p(t) > 0$ for any $t \in [r_k^i, r_k^{ii}]$ and $k \in N$.

Reasoning similarly, we obtain that there exists a sequence of segments $\{[l_k^i, l_k^{ii}]\}_{k=1}^\infty$, $l_1^{ii} > l_1^i > l_2^{ii} > l_2^i > \dots$, $\lim_{k \rightarrow +\infty} l_k^i = l$, such that $p(t) > 0$ for any $t \in [l_k^i, l_k^{ii}]$ and $k \in N$. We suppose $r_1^i \geq b$, $l_1^{ii} \leq a$ (the numbers a and b are from condition (c) of Theorem 1). Set

$$f_k(x) := \int_{l_k^i}^{r_k^{ii}} \exp(xt)p(t)dt, \quad k \in N. \quad (5)$$

Let us show that $f_{k_0}(x) > 0$ for some $k_0 \in N$ and any $x \in R$. We have

$$f_k(x) \geq \int_{l_1^i}^{r_1^{ii}} \exp(xt)p(t)dt = f_1(x), \quad x \in R. \quad (6)$$

Let $x \geq 0$, then it follows from (6) that

$$\begin{aligned} f_k(x) &\geq \int_{l_1^i}^{r_1^{ii}} \exp(xt)p(t)dt \geq \int_{(r_1^i+r_1^{ii})/2}^{r_1^{ii}} \exp(xt)p(t)dt - \\ &- \int_{l_1^i}^{r_1^i} \exp(xt)|p(t)|dt \geq K_1 \exp((r_1^i + r_1^{ii})x/2) - K_2 \exp(r_1^i x), \end{aligned} \quad (7)$$

where $K_1 > 0$ and $K_2 > 0$ are independent of x and k . Thus there exists a positive number R_1 such that $f_k(x) > 0$ for any $x \geq R_1$ and $k \in N$. Reasoning similarly for $x \leq 0$ we obtain

$$f_k(x) \geq K_3 \exp((l_1^i + l_1^{\prime\prime})x/2) - K_4 \exp(l_1^{\prime\prime}x), \quad (8)$$

where $K_3 > 0$ and $K_4 > 0$ do not depend on x and k . So there exists a negative number R_2 such that $f_k(x) > 0$ for any $x \leq R_2$ and $k \in N$. Thus $f_k(x) > 0$ for any $k \in N$ and for any $x \in (-\infty; R_2] \cup [R_1; +\infty)$. It remains to show that there exists $k \in N$ such that $f_k(x) > 0$, when $x \in [R_2; R_1]$. It follows from conditions (a), (b) of Theorem 1 that $f_k(x) \rightarrow f(x)$, $k \rightarrow +\infty$ uniformly on any compact set in R . It follows from condition (e) of Theorem 1 that there exists $k_0 \in N$ such that for any $x \in [R_2; R_1]$ the inequality $f_{k_0}(x) > 0$ holds. Therefore we have $f_{k_0}(x) > 0$ for any $x \in R$.

Denote

$$p_1(t) = p(t)\chi[l, l_{k_0}^i](t), \quad (9)$$

$$p_2(t) = p(t)\chi[l_{k_0}^i, r_{k_0}^{\prime\prime}](t), \quad (10)$$

$$p_3(t) = p(t)\chi[r_{k_0}^{\prime\prime}, r](t), \quad (11)$$

where $\chi[m, n](t)$ means the characteristic function of segment $[m, n]$:

$$\chi[m, n](t) = \begin{cases} 1, & t \in [m, n], \\ 0, & t \in (-\infty; m) \cup (n; +\infty). \end{cases} \quad (12)$$

Then

$$p(t) = p_1(t) + p_2(t) + p_3(t), \quad (13)$$

where

$$p_1(t) \geq 0, \quad p_3(t) \geq 0. \quad (14)$$

It is clear that

$$\exp(\lambda e^{\varepsilon x}) = \sum_{k=0}^{\infty} \frac{\lambda^k \exp(k\varepsilon x)}{k!} = \int_0^{\infty} \exp(xt) d\Phi_{\lambda, \varepsilon}(t) \quad (15)$$

where $\Phi_{\lambda, \varepsilon}(t)$ is the step function with jump $\lambda^k/k!$ at the point εk , $k = 0, 1, 2, \dots$. Thus the function $F_{\lambda, \varepsilon}(x) = f(x) \exp(\lambda e^{\varepsilon x})$ has the representation

$$F_{\lambda, \varepsilon}(x) = \int_0^{\infty} \exp(xt) q_{\lambda, \varepsilon}(t) dt, \quad (16)$$

where

$$q_{\lambda, \varepsilon}(t) = (p * \Phi_{\lambda, \varepsilon})(t) = (p_1 * \Phi_{\lambda, \varepsilon})(t) + (p_2 * \Phi_{\lambda, \varepsilon})(t) + (p_3 * \Phi_{\lambda, \varepsilon})(t). \quad (17)$$

For $\varepsilon > 0$, $\lambda > 0$ and $t > 0$ we have

$$(p_1 * \Phi_{\lambda,\varepsilon})(t) \geq 0, \quad (p_3 * \Phi_{\lambda,\varepsilon})(t) \geq 0. \quad (18)$$

Thus we have to prove that $(p_2 * \Phi_{\lambda,\varepsilon})(t) \geq 0$ for $0 < \varepsilon \leq \varepsilon_0$ and $\lambda \geq \lambda_\varepsilon$. This follows from Theorem 2 (because the function p_2 satisfies all conditions of this theorem).

So, Theorem 1 will be proved if we prove Theorem 2.

3 The proof of Theorem 2

3.1

Without loss of generality we can suppose that $[a, b] = [0, 1]$. It follows from conditions (a), (b) that there exists $\Delta \in (0, 1)$ such that

$$p(t) > 0, \quad \text{when } t \in [0, \Delta] \cup [1 - \Delta, 1]. \quad (19)$$

We shall assume that p is defined at every point of the real axis by putting $p(t) = 0$, when $t \in (-\infty, 0) \cup (1; +\infty)$.

We have

$$F_{\lambda,\varepsilon}(x) = f(x) \exp(\lambda e^{\varepsilon x}) = \int_0^\infty \exp(xt) q_{\lambda,\varepsilon}(t) dt, \quad (20)$$

where

$$q_{\lambda,\varepsilon}(t) = (p * \Phi_{\lambda,\varepsilon})(t). \quad (21)$$

Let us compute $q_{\lambda,\varepsilon}$. By definition of $\Phi_{\lambda,\varepsilon}(t)$ (see (15)) we have

$$q_{\lambda,\varepsilon}(t) = \int_{t-1}^t p(t-u) d\Phi_{\lambda,\varepsilon}(u). \quad (22)$$

By virtue of (22) it is clear that

$$q_{\lambda,\varepsilon}(t) = 0, \quad \text{when } t \leq 0 \quad (23)$$

moreover by (19) we have

$$q_{\lambda,\varepsilon}(t) \geq 0, \quad \text{when } t \in [0, \Delta]. \quad (24)$$

We are interested in the form of the function $q_{\lambda,\varepsilon}(t)$ for $t \geq \Delta$. Let us fix such t .

We will use the following notations: $m = m(\varepsilon, t) := [t/\varepsilon]$, $\delta = \delta(\varepsilon, t) := t - [t/\varepsilon]\varepsilon$, $n = n(\varepsilon, t) := [(1 - \delta)/\varepsilon]^*$, where $[a]$ denotes the largest integer $< a$, $[a]^*$ denotes the smallest integer $> a$.

Note that

$$n(\varepsilon, t) \leq [\varepsilon^{-1}] + 1. \quad (25)$$

We have from (15) and (22)

$$\begin{aligned} q_{\lambda, \varepsilon}(t) &= \int_{t-1}^t p(t-u) d\Phi_{\lambda, \varepsilon}(u) = \\ &= \frac{p(\delta)\lambda^m}{m!} + \frac{p(\delta+\varepsilon)\lambda^{m-1}}{(m-1)!} + \dots + \\ &\quad + \frac{p(\delta+(n-1)\varepsilon)\lambda^{m-n+1}}{(m-n+1)!} = \\ &= \frac{\lambda^m}{m!} \left(p(\delta) + p(\delta+\varepsilon)\frac{m}{\lambda} + p(\delta+2\varepsilon)\frac{m}{\lambda} \left(\frac{m}{\lambda} - \frac{1}{\lambda} \right) + \dots + \right. \\ &\quad \left. + p(\delta+(n-1)\varepsilon)\frac{m}{\lambda} \left(\frac{m}{\lambda} - \frac{1}{\lambda} \right) \dots \left(\frac{m}{\lambda} - \frac{n-2}{\lambda} \right) \right). \quad (26) \end{aligned}$$

3.2

Let us introduce the following system of polynomials on y :

$$\begin{aligned} Q_n(y, \mu, \beta) &= p(\mu) + p(\mu+\varepsilon)y + p(\mu+2\varepsilon)y(y-\beta) + \dots + \\ &\quad + p(\mu+(n-1)\varepsilon)y(y-\beta) \dots (y-(n-2)\beta), \quad (27) \end{aligned}$$

where

$$0 \leq \mu \leq \varepsilon, \quad 0 \leq \beta \leq 1, \quad 0 \leq \varepsilon \leq 1, \quad n \leq [\varepsilon^{-1}] + 1, \quad (28)$$

(compare with (25) and (26)).

From (26) we see that

$$q_{\lambda, \varepsilon}(t) = \frac{\lambda^m}{m!} Q_n \left(\frac{m}{\lambda}, \delta, \frac{1}{\lambda} \right), t \geq 0. \quad (29)$$

We shall study the system of the polynomials $Q_n(y, \mu, \beta)$. Let $\varepsilon > 0$ be fixed and sufficiently small. (The value of $\varepsilon > 0$ will be chosen later). By (27) the degrees of $Q_n(y, \mu, \beta)$ are bounded, when ε is fixed. Let us show that under these conditions the polynomials $Q_n(y, \mu, \beta)$ have the common boundary of positive zeros for any $\mu \in [0, \varepsilon]$ and $\beta \in [0, 1]$. We shall use the following well-known theorem.

Theorem B *The polynomial $P(z) = a_0 + a_1z + \dots + a_nz^n$ has no zeros when $|z| \geq R$, where*

$$R = 1 + (1/a_n) \max \{|a_j| : 0 \leq j \leq n-1\}. \quad (30)$$

The number $p(\mu + (n - 1)\varepsilon)$ is the leading coefficient of the polynomial $Q_n(y, \mu, \beta)$. Let $\varepsilon > 0$ be a fixed number, $\varepsilon < \Delta/4$ (see (19)). If $\mu \in [0, \varepsilon]$, $\beta \in [0, 1]$, $n \leq [\varepsilon^{-1}] + 1$ we have

$$p(\mu + (n - 1)\varepsilon) \geq \min \{p(t) : t \in [1 - \Delta, 1]\} > 0. \quad (31)$$

It is clear that the rest of the coefficients of $Q_n(y, \mu, \beta)$ are bounded from above by the constant depending only on $\max \{|p(t)| : 0 \leq t \leq 1\}$ if ε is fixed, $\varepsilon \in (0, \Delta/4]$, $\mu \in [0, \varepsilon]$, $\beta \in [0, 1]$. Thus for fixed $\varepsilon \in (0, \Delta/4]$ there exists a real number $R_0 > 1$ not depending on β such that

$$Q_n(y, \mu, \beta) > 0, \quad y \geq R_0 \quad (32)$$

3.3

Let $\varepsilon > 0$ be a fixed small number. If it is remembered that p is a continuous function, the number $n = n(\varepsilon, t)$ satisfies (25), we obtain the following :

$$\begin{aligned} \lim_{\beta \rightarrow 0} Q_n(y, \mu, \beta) &= p(\mu) + p(\mu + \varepsilon)y + p(\mu + 2\varepsilon)y^2 + \dots + \\ &+ p(\mu + (n - 1)\varepsilon)y^{n-1} =: Q_n(y, \mu), \end{aligned} \quad (33)$$

where the limit is uniform with respect to $\mu \in [0, 1]$ and $y \in C$, for every compact $C \subset C$.

Lemma 1 *There exist positive numbers ε_0 and ν_0 such that*

$$Q_n(y, \mu) \geq \nu_0 > 0 \quad (34)$$

for any $\varepsilon \in (0, \varepsilon_0]$, $\mu \in [0, \varepsilon]$ and $y \geq 0$.

Before proving Lemma 1 we show that Theorem 2 follows from Lemma 1.

3.4

Let us assume that Lemma 1 is true. We choose and fix a positive number $\varepsilon \in (0, \varepsilon_0]$ (ε_0 is the same as in Lemma 1). Consider the system of polynomials $Q_n(y, \mu, \beta)$ on the segment $[0, R_0]$, where R_0 is from (32). Since ε_0 is fixed, by (25) we have that the degrees of these polynomials are bounded by the number $[\varepsilon^{-1}] + 1$. By (34) and uniform convergence $Q_n(y, \mu, \beta)$ to $Q_n(y, \mu)$ with respect to $y \in [0, R_0]$ there exists a positive number β_0 such that

$$Q_n(y, \mu, \beta) \geq \nu_0/2 > 0 \quad (35)$$

for any $\beta \in (0, \beta_0]$, $y \in [0, R_0]$ and $\mu \in [0, \varepsilon]$. By (32) and (35) there exists a positive number ε_0 and for any $\varepsilon \in (0, \varepsilon_0]$ we can choose $\beta_0 > 0$ such that for any $\beta \in (0, \beta_0]$, $y \geq 0$ the inequality $Q_n(y, \mu, \beta) > 0$ is valid. Putting $\lambda_\varepsilon = (1/\beta_0)$ and recalling (29) we obtain the statement of Theorem 2.

Thus Theorem 2 will be proved if we prove Lemma 1.

3.5

To prove Lemma 1 we have to find more precise boundary of positive zeros of the polynomial $Q_n(y, \mu, \beta)$ than that given by Theorem B.

Lemma 2 *Let $F(t) = a_0 + a_1t + \dots + a_{n-1}t^{n-1}$ be a polynomial with real coefficients. Suppose that for some $q \in N$ such that $n/2 \leq q < n$ the inequalities*

$$a_q > 0, \quad a_{q+1} > 0, \quad \dots, \quad a_{n-1} > 0 \quad (36)$$

are valid. Let

$$\max \{|a_j| : 0 \leq j \leq q-1\} \leq B, \quad \min \{a_j : q \leq j \leq n-1\} \geq b. \quad (37)$$

Then $F(t) > 0$ for

$$t \geq \max \left\{ 1, \left(\frac{Bq}{b(n-q)} \right)^{1/(n-q)} \right\}. \quad (38)$$

3.6 Proof of Lemma 2

We have (for $t \geq 0$)

$$\begin{aligned} F(t) &= a_{n-1}t^{n-1} + \dots + a_q t^q + a_{q-1}t^{q-1} + \dots + a_0 \geq \\ &\geq b(t^{n-1} + t^{n-2} + \dots + t^q) - B(t^{q-1} + t^{q-2} + \dots + 1) = \\ &= bt^q \frac{t^{n-q} - 1}{t-1} - B \frac{t^q - 1}{t-1} = \\ &= b \frac{t^{n-q} - 1}{t-1} \left[t^q - \frac{B}{b} \frac{t^q - 1}{t^{n-q} - 1} \right]. \end{aligned} \quad (39)$$

It is not hard to see that under the conditions of Lemma 2

$$\frac{t^q - 1}{t^{n-q} - 1} \leq \frac{q}{n-q} t^{2q-n} \quad (40)$$

for $t \geq 1$. (39) and (40) yield

$$F(t) \geq b \frac{t^{n-q} - 1}{t-1} \left[t^q - Bq \frac{t^{2q-n}}{b(n-q)} \right]. \quad (41)$$

Thus, if $t \geq 1$ and satisfies the condition

$$t^q - Bq \frac{t^{2q-n}}{b(n-q)} > 0, \quad (42)$$

then $F(t) > 0$. It is easily verified that (42) implies the statement of Lemma 2.

3.7 Proof of Lemma 1

We shall find the upper and lower boundaries of the positive zeros of the polynomial $Q_n(y, \mu)$ with the help of Lemma 2. Put

$$B = \max \{|p(t)| : 0 \leq t \leq 1\}; \quad b = \min \{p(t) : t \in [0, \Delta] \cup [1 - \Delta, 1]\} > 0. \quad (43)$$

The number $p(\mu + k\varepsilon)$ (see (33)) is the coefficient of the polynomial $Q_n(y, \mu)$ of y^k . Under the condition we have $p(t) > 0$ for $t \in [0, \Delta] \cup [1 - \Delta, 1]$. We apply Lemma 2 to the polynomials

$$F_1(y) = Q_n(y, \mu) - b/2 \quad (44)$$

and

$$F_2(y) = y^{n-1} (Q_n(1/y, \mu) - b/2) = (p(\mu) - b/2) y^{n-1} + p(\mu + \varepsilon) y^{n-2} + p(\mu + 2\varepsilon) y^{n-3} + \dots + p(\mu + (n-1)\varepsilon). \quad (45)$$

Put

$$\varepsilon_1 = \min \{1/4, \Delta/4\}. \quad (46)$$

We shall choose $\varepsilon_0 < \varepsilon_1$ (recall that $0 < \mu \leq \varepsilon < \varepsilon_0$). Let us estimate the positive integer numbers n and q from Lemma 2 for polynomials $F_1(y)$ and $F_2(y)$. Note that the number n from Lemma 2 is the number of all coefficients of the polynomial F and the number q is the number of the positive ‘‘leading’’ coefficients. By (19) ‘‘leading’’ coefficients of polynomial $F_1(y)$ will be positive, if the index k satisfies $1 - \Delta < \mu + k\varepsilon < 1$. So for polynomial $F_1(y)$ the number of positive ‘‘leading’’ coefficients is not less than $[\Delta/\varepsilon] - 1$, and by (25) the general number of the coefficients are not exceeding $[1/\varepsilon] + 1$. Applying Lemma 2 to the polynomial $F_1(y)$ we obtain

$$F_1(y) > 0, \quad \text{when} \quad y \geq \max \left\{ 1, \left(\frac{2B([\Delta/\varepsilon] + 1)}{b([\Delta/\varepsilon] - 1)} \right)^{1/([\Delta/\varepsilon] - 1)} \right\}. \quad (47)$$

Thus, there exists a constant $C_1 > 0$ not depending on μ and ε such that

$$Q_n(y, \mu) > b/2, \quad \text{when} \quad y \geq \exp(C_1\varepsilon). \quad (48)$$

Reasoning similarly for polynomial $F_2(y)$ we obtain that there exists a constant $C_2 > 0$ not depending on μ and ε such that

$$Q_n(y, \mu) > b/2, \quad \text{when} \quad 0 \leq y \leq \exp(-C_2\varepsilon). \quad (49)$$

It remains to prove that the polynomials $Q_n(y, \mu)$ are bounded from below by a positive constant, when $\exp(-C_2\varepsilon) \leq y \leq \exp(C_1\varepsilon)$. We apply the substitution $y = \exp(x\varepsilon)$. This substitution is correct, because we consider the system of polynomials $Q_n(y, \mu)$ for $y > 0$. Put

$$q_n(x, \mu) = Q_n(e^{x\varepsilon}, \mu) = p(\mu) + p(\mu + \varepsilon)e^{x\varepsilon} + p(\mu + 2\varepsilon)e^{2x\varepsilon} + \dots + p(\mu + (n-1)\varepsilon)e^{(n-1)x\varepsilon}. \quad (50)$$

By virtue (48) and (49) we have

$$q_n(x, \mu) > b/2, \quad \text{when} \quad |x| \geq C = \max\{C_1, C_2\}, \quad (51)$$

where C is independent of μ and ε . We shall consider the functions $q_n(x, \mu)$ for $|x| \leq C$. By conditions (a), (c) of Theorem 2 there exists a positive number a_1 such that

$$f(x) = \int_0^1 \exp(xt)p(t)dt \geq a_1 \quad (52)$$

for any $|x| \leq C$. Note that

$$\begin{aligned} \varepsilon \exp(x\mu)q_n(x, \mu) &= \varepsilon \left(p(\mu) + \right. \\ &\quad \left. + p(\mu + \varepsilon)e^{x(\varepsilon+\mu)} + p(\mu + 2\varepsilon)e^{x(2\varepsilon+\mu)} + \dots + \right. \\ &\quad \left. + p(\mu + (n-1)\varepsilon)e^{x(n-1)\varepsilon+\mu} \right) \rightarrow \\ &\rightarrow \int_0^1 \exp(xt)p(t)dt \geq a_1, \quad \varepsilon \rightarrow 0, \end{aligned} \quad (53)$$

uniformly with respect to $x \in [-C, C]$. Thus, there exists a positive number $\varepsilon_0(C)$ such that for any $\varepsilon \in (0, \varepsilon_0(C))$ we have

$$\varepsilon \exp(x\mu)q_n(x, \mu) \geq a_1/2, \quad \text{when} \quad |x| \leq C, \quad \mu \in [0, \varepsilon]. \quad (54)$$

From (54) we obtain that

$$q_n(x, \mu) > a_1 \exp(-C\varepsilon_0)/(2\varepsilon_0), \quad \text{when} \quad |x| \leq C. \quad (55)$$

Put

$$\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_0(C)\}, \quad \nu_0 = \min\{b/2, a_1 \exp(-C\varepsilon_0)/(2\varepsilon_0)\}. \quad (56)$$

Then for any $\varepsilon \in (0; \varepsilon_0]$ and $x \in [-C, C]$ the inequality

$$q_n(x, \mu) \geq \nu_0 > 0 \quad (57)$$

holds. By (51) and (57) we obtain $q_n(x, \mu) \geq \nu_0 > 0$ for any $\varepsilon \in (0; \varepsilon_0]$, $\mu \in (0; \varepsilon]$, $x \in R$. Recalling that $Q_n(y, \mu) = q_n(\varepsilon^{-1} \ln y, \mu)$, we obtain

$$Q_n(y, \mu) \geq \nu_0 > 0 \quad (58)$$

for any $\varepsilon \in (0; \varepsilon_0]$, $\mu \in (0; \varepsilon]$, $y \geq 0$. Thus Lemma 1 is proved. This completes the proof of Theorem 2 and Theorem 1.

Remark. The function p in Theorems 1 and 2 can be viewed as a density of some real-valued measure m_p . Theorems 1 and 2 can be reformulated in such a way. Let m_p be a real-valued measure with a continuous density supported by the positive half-axis or a segment. Let the Laplace transform of this measure $L_{m_p}(x) > 0$ for all $x \in R$ and m_p is positive in some neighborhoods of the

endpoints of its support. Then for some $\varepsilon > 0$ and $\lambda_\varepsilon > 0$ the convolution $m_p * \mu_{\lambda,\varepsilon} \geq 0$ (where $L_{\mu_{\lambda,\varepsilon}}(x) = \exp(\lambda e^{\varepsilon x})$). A question can be asked: is such statement true for any measure supported by half-axis or a segment which is positive in some neighborhoods of the endpoints of its support and with positive Laplace transform? The answer is negative. Let $\alpha, \beta \in (0, 1)$ be irrational non-measurable numbers. Let $m := D_0 - \varepsilon D_\alpha - \varepsilon D_\beta + D_1$, where D_x is the Dirac measure at the point x and $\varepsilon > 0$. It is clear that if ε is sufficiently small then $L_m(x) = \int_0^1 \exp(xt) dm(t) > 0$ for all real x . If the convolution $m * \mu_{\lambda,\varepsilon}$ is nonnegative in the point α then $\varepsilon = \frac{\alpha}{k}, k \in \mathbb{N}$, and if $m * \mu_{\lambda,\varepsilon}$ is nonnegative in the point β then $\varepsilon = \frac{\beta}{l}, l \in \mathbb{N}$. So for any choice of ε, λ the convolution $m * \mu_{\lambda,\varepsilon}$ can not be nonnegative.

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