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ON A CONJECTURE OF YU. V. LINNIK

A. M. VISHNYAKOVA, I. V. OSTROVSKII, AND A. M. ULANOVSKII

ABSTRACT. A survey is given of work concerning Linnik's 1960 conjecture on the growth of entire characteristic functions (Fourier transforms) of probability measures. A proof of this conjecture is presented that is substantially shorter and more elementary than the known proofs. With the help of the same idea, new facts are established about the growth of entire characteristic functions with restrictions on the arguments of the zeros.

§1. Results connected with the Linnik conjecture

An entire function $\varphi\colon \mathbb{C} \to \mathbb{C}$ is called an *entire characteristic function* (ECF) if its restriction to $\mathbb{R}$ is a characteristic function in the sense common in probability theory, i.e., if the representation

$$\varphi(z) = \int_{\mathbb{R}} e^{izu} P(du) \tag{1.1}$$

holds for $z \in \mathbb{R}$, where P is a probability measure on $\mathbb{R}$. It can be shown ([1], proof of Theorem 2.2.2) that if φ is an ECF, then for all $z \in \mathbb{C}$ the integral in (1.1) converges absolutely, and hence the representation (1.1) is preserved. Thus, an ECF can be defined to be a function representable for all $z \in \mathbb{C}$ in the form (1.1), where the integral converges absolutely. The latter means that P decreases sufficiently rapidly at infinity. Therefore, ECF's are encountered fairly often in probability theory; as examples we present the ECF's

$$\exp\{-\gamma z^2 + i\beta z\}, \quad \gamma \geq 0,\ \beta \in \mathbb{R}, \qquad \text{and} \qquad \exp\{\lambda(e^{iz} - 1)\}, \quad \lambda \geq 0,$$

corresponding to Gaussian and Poisson measures (in particular, to degenerate measures).

It is known ([1], Chapter II, §4) that the growth of a nonconstant ECF φ satisfies only the single restriction

$$\varliminf_{r\to\infty} r^{-1} \log M(r, \varphi) > 0 \qquad (M(r, \varphi) = \max\{|\varphi(re^{i\theta})|\colon 0 \leq \theta \leq 2\pi\}).$$

Marcinkiewicz [2] observed that the situation changes essentially if the ECF has "few" zeros. It was established in [2] that if an ECF φ has finite order $\rho[\varphi]$, and the exponent of convergence of the zeros is strictly less than $\rho[\varphi]$, then $\rho[\varphi] \leq 2$. As a corollary, a result was obtained in [2] that is now known as the Marcinkiewicz theorem and has many applications in probability theory and mathematical statistics (see, for example, [3]).

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MARCINKIEWICZ THEOREM. *If an ECF φ of finite order does not have zeros, then it is the CF of a Gaussian measure, i.e., $\varphi(z) = \exp\{-\gamma z^2 + i\beta z\}$, $\gamma \geq 0$, $\beta \in \mathbb{R}$.*

By Hadamard's theorem on representation of an entire function of finite order, this assertion can be reformulated as follows. If an ECF φ has the form

$$\varphi(z) = \exp f(z), \tag{1.2}$$

where f is a polynomial, then $f(z) = -\gamma z^2 + i\beta z$, $\gamma \geq 0$, $\beta \in \mathbb{R}$.

We present a proof of this fact. It is based on the following property of ECF's, which follows directly from the validity of (1.1) for all $z \in \mathbb{C}$ and is called the "ridge property":

$$|\varphi(z)| \leq \varphi(i \operatorname{Im} z), \qquad z \in \mathbb{C}. \tag{1.3}$$

Let $f(z) = a_n z^n + \cdots + a_0$, $a_n \neq 0$. It follows from (1.2) and (1.3) that

$$\operatorname{Re}(a_n z^n + \cdots + a_0) \leq |a_n (i \operatorname{Im} z)^n + \cdots + a_0|. \tag{1.4}$$

Let $a_n = |a_n| e^{i\alpha_n}$ and $z = re^{i\varphi}$. On the rays $\varphi = \varphi_k = (-\alpha_n + 2k\pi)/n$, $k = 0, \dots, n-1$, the left-hand and right-hand sides of (1.4) are respectively equal to $|a_n| r^n + O(r^{n-1})$ and $|a_n| r^n |\sin \varphi_k|^n + O(r^{n-1})$, $r \to \infty$. Therefore, $\varphi_k \equiv \pi/2 \pmod{\pi}$, $k = 0, 1, \dots, n-1$, which is possible only for $n \leq 2$; and in the case $n = 2$ it must also happen that $\alpha_2 \equiv \pi \pmod{2\pi}$, i.e., $a_2 = -\gamma$, $\gamma > 0$. Since $\varphi(0) = 1$, it can be assumed that $a_0 = 0$. And since $\varphi(z) = 1 + a_1 z + O(z^2)$, $z \to 0$, we see from (1.3) that $a_1 = i\beta$, $\beta \in \mathbb{R}$.

It is not hard to see that the assumption of finite order in the Marcinkiewicz theorem (which is equivalent to the assumption that the entire function f in (1.2) is a polynomial) cannot be discarded. An elementary example is supplied by the ECF of a Poisson measure: $\varphi(z) = \exp\{\lambda(e^{iz} - 1)\}$, $\lambda \geq 0$. One can also construct ([1], Russian p. 64, English p. 45) examples of ECF's without zeros and of an arbitrary more rapid growth. In 1960 Linnik conjectured that there is no ECF without zeros having growth intermediate between the growth of the ECF of a Gaussian measure and the growth of the ECF of a Poisson measure. More precisely, this conjecture ([4], Chapter XIII, §2) amounts to the assertion that the assumption of finiteness of order in the Marcinkiewicz theorem can be replaced by the assumption that the entire function f in (1.2) satisfies the condition

$$\lim_{r \to \infty} r^{-1} \log M(r, f) = 0. \tag{1.5}$$

Linnik's conjecture was proved by one of the authors ([5], [6]). The proof was very complicated, however it gave an essentially more general result. This proof was based on the fact, noticed as far back as 1914 by Wiman and Valiron, that any entire function f behaves in a sufficiently small neighborhood $\Delta(\zeta)$ of a point ζ of maximum modulus (i.e., a point ζ where $|f(\zeta)| = M(|\zeta|, f)$) "approximately" like a polynomial whose degree $n = n(\zeta, f)$ increases unboundedly as $\zeta \to \infty$. More precisely,

$$f(z) = (z/\zeta)^n f(\zeta)(1 + \omega(z, \zeta)), \tag{1.6}$$

where the remainder term ω tends to 0 for $z \in \Delta(\zeta)$, $\zeta \to \infty$.

If the size of the neighborhood $\Delta(\zeta)$ in which (1.6) holds with a satisfactory estimate for ω is sufficiently large, then it is possible to carry through an argument close to the proof of the Marcinkiewicz theorem. To do this we use the consequence $\operatorname{Re} f(z) \leq |f(i \operatorname{Im} z)|$ of the ridge property (this inequality was written in the form (1.4) in the case of the Marcinkiewicz theorem), setting in it $z = z_k = \zeta e^{i\beta_k}$, $\beta_k = (-\arg f(\zeta) + 2k\pi)/n$, $k \in \mathbb{Z}$. By (1.6), $\operatorname{Re} f(z_k) \geq M(|\zeta|, f)(1 - |\omega(z_k, \zeta)|)$. On the other hand, it is obvious that $|f(i \operatorname{Im} z_k)| \leq M(|\zeta| |\sin(\beta_k + \arg \zeta)|, f)$. If the size of the neighborhood $\Delta(\zeta)$ permits us to vary k within sufficiently broad limits (so that $|\sin(\beta_k + \arg \zeta)|$ is essentially less than 1), then the relation

$$M(|\zeta|, f)(1 - |\omega|) \leq M(|\zeta| |\sin(\beta_k + \arg \zeta)|, f)$$

turns out to be impossible.

The main difficulty in the realization of this idea was the proof that (1.6) is valid with a satisfactory estimate of the remainder term ω in a neighborhood $\Delta(\zeta)$ of sufficiently large size—essentially larger than in the classical results of Wiman and Valiron. This proof [6] is fairly cumbersome and is based on a refinement of the method used for getting relations of the type (1.6) in the article [7] of Macintyre.

The statement of the basic result in [6] is as follows.

THEOREM A [6]. *Let Q be an arbitrary nonconstant entire function, and f an entire function satisfying condition* (1.5). *If $\varphi(x) = Q(f(z))$ has the ridge property, then f is a polynomial of degree at most* 2.

A subsequent improvement of the method described led to the following result.

THEOREM B [8]. *The assertion of Theorem* A *remains in force if condition* (1.5) *is replaced by the weaker condition*

$$\varliminf_{r\to\infty} r^{-1} \log M(r, f) = 0. \tag{1.7}$$

The validity of Linnik's conjecture follows from Theorem A for $Q(z) = e^z$; by Theorem B, condition (1.5) can be replaced by (1.7).

Theorem A shows also that the assumption that φ is an ECF can be replaced by the presence of the ridge property for φ. The entire functions φ with $\varphi(0) = 1$ having this property form ([1], Russian pp. 47–50, English pp. 31–34) a class broader than the class of ECF's. They are called *entire ridge functions* (ERF's). In what follows we shall consider ERF's instead of ECF's. Note that only the fact that φ is an ERF was used in the above proof of the Marcinkiewicz theorem. It can be shown [9] that any ERF φ admits a representation $\varphi = \varphi_1/\varphi_2$, where φ_1 and φ_2 are ECF's, and φ_2 does not have zeros. This fact, fairly complicated to prove, will not be needed below.

The authors of the present article found a brief and elementary proof of the Linnik conjecture, based on an idea not connected with those used in [5], [6], and [8]. Unfortunately, we do not see the possibility of using this idea to prove Theorem A. However, it does enable us to get a new result not contained in Theorem A nor Theorem B. This result is stronger than the assertion that Linnik's conjecture is valid, and is connected with analogues of the Marcinkiewicz theorem for ERF's *with restrictions on the arguments of the zeros.*

The first such analogue was apparently obtained in [10], where the following result was proved: If an ERF φ of finite order has only real zeros, then $\rho[\varphi] \leq 2$. The requirement that the zeros be real here replaces the requirements in the Marcinkiewicz results [2] that there be no zeros or that their exponent of convergence be strictly less than $\rho[\varphi]$. In [11] the result of [10] was strengthened as follows.

THEOREM C [11]. *Suppose that an ERF φ of finite order does not have zeros in the domain*

$$G_\alpha = \{z\colon |\arg z - \pi/2| < \alpha\} \cup \{z\colon |\arg z + \pi/2| < \alpha\}, \qquad 0 < \alpha \le \pi/2.$$

Then $\rho[\varphi] < \pi/\alpha$ for $0 < \alpha < \pi/4$ and $\rho[\varphi] \le 2$ for $\pi/4 \le \alpha \le \pi/2$.

We get the result in [10] for $\alpha = \pi/2$.

In connection with the Linnik conjecture (and condition (1.7) in Theorem B) the question naturally arises as to whether the condition of finiteness of order in Theorem C can be replaced by the condition

$$\varliminf_{r\to\infty} r^{-1} \log\log M(r, \varphi) = 0. \tag{1.8}$$

A positive answer was obtained in [12] in the case $\alpha = \pi/2$.

The main result in the present article is the following theorem.

THEOREM 1. *Suppose that an ERF φ satisfies condition* (1.8) *and does not have zeros in G_α. Then $\rho[\varphi] < \infty$.*

An immediate consequence of Theorem 1 and Theorem C is a positive answer to the indicated question for all α with $0 < \alpha \le \pi/2$:

THEOREM 2. *The assertion of Theorem C remains in force if the condition of finiteness of order is replaced by condition* (1.8).

If the ERF φ has the form $\varphi = \exp f$, where f satisfies condition (1.7), then condition (1.8) holds, and $\rho[\varphi] < \infty$ by Theorem 1. Using the Marcinkiewicz theorem, we conclude that φ is the CF of a Gaussian measure, i.e., the Linnik conjecture is valid.

REMARK 1. The question arises of the sharpness of the estimates for $\rho[\varphi]$ given in Theorem C (and Theorem 2). The example of the CF of a Gaussian measure shows that the estimate $\rho[\varphi] \le 2$ $(\pi/4 \le \alpha \le \pi/2)$ cannot be improved even in the class of ECF's, which is smaller than the class of ERF's. It was observed in [11] that for $0 < \alpha < \pi/4$ there exists an ECF φ_α not having zeros in G_α and such that $\rho[\varphi_\alpha] = \gamma(\alpha)$, where $\gamma(\alpha) < \pi/\alpha$ is defined as follows: $\gamma(\alpha) = \pi/(2\alpha)$ if $\pi/6 \le \alpha < \pi/4$; $\gamma(\alpha)$ is the root of the equation $\cos\gamma(\alpha + \pi/\gamma) = -\cos\gamma\alpha$ lying in $(\pi/(2\alpha), \pi/\alpha)$ if $0 < \alpha < \pi/6$. Note that $\gamma(\alpha) = \pi/\alpha - 2\sqrt{\pi/\alpha}(1 + o(1))$, $\alpha \to 0$. Recently Vishnyakova and Fryntov, independently and using different methods, proved the estimate $\rho[\varphi] \le \gamma(\alpha)$, $0 < \alpha < \pi/4$, under the conditions of Theorem C. Their papers will be printed in the collection *Analytic questions in probability theory*, published by Naukova Dumka in Kiev. Obviously, it follows from their result and Theorem 1 that the estimate $\rho[\varphi] \le \gamma(\alpha)$, $0 < \alpha < \pi/4$, remains in force also under the conditions of Theorem 2.

§2. Proof of Theorem 1

We list the needed properties of ERF's φ: a) $\varphi(iy) > 0$ for $y \in \mathbb{R}$; b) the function $\log\varphi(iy)$ is convex on $\mathbb{R}$; c) $M(r, \varphi) = \max\{\varphi(ir), \varphi(-ir)\}$, $r > 0$. Property a) follows immediately from (1.3), and property c) is obtained from (1.3) with the help of the maximum modulus principle. To get property b) it suffices to observe that the function $\log|\varphi(x+iy)|$ is harmonic in a neigbhborhood of the imaginary axis, while $(\partial/\partial x)^2\log|\varphi| \le 0$ on this axis in view of (1.3).

We also use the following well-known fact.

THEOREM (Carathéodory inequality; [13], Chapter I, §6, Theorem 8). *If a function g is analytic in the disk $\{\zeta : |\zeta| < 1\}$ and satisfies there the condition $\operatorname{Re} g(\zeta) \le A < \infty$, then*

$$|g(\zeta)| \le \frac{2|\zeta|}{1-|\zeta|}\{A - \operatorname{Re} g(0)\} + |g(0)|, \qquad |\zeta| < 1. \tag{2.1}$$

Suppose that an ERF satisfies the conditions of Theorem 1. It can be assumed without loss of generality that $\alpha < \pi/2$. Denote by f the analytic branch of $\log\varphi$ singled out by the condition $f(0) = 0$ on the domain $G_\alpha \cup \{z: |z| < \varepsilon\}$ (where ε is sufficiently small). By the properties of ERF's noted above, we have: a_1) $\operatorname{Im} f(iy) = 0$ for $y \in \mathbb{R}$; b_1) the function $f(iy)$ is convex on $\mathbb{R}$; and c_1) $\log M(r, \varphi) = \max\{f(ir), f(-ir)\}$, $r > 0$. Moreover, it follows directly from (1.3) that

$$\operatorname{Re} f(z) \le f(i\operatorname{Im} z), \qquad z \in G_\alpha \cup \{z: |z| < \varepsilon\}. \tag{2.2}$$

From condition (1.8),

$$\varliminf_{r\to\infty} r^{-1}\log\{f(\pm ir)\}^+ \le 0. \tag{2.3}$$

Theorem 1 will be proved if it is established that

$$\varlimsup_{r\to\infty} (\log r)^{-1}\log\{f(\pm ir)\}^+ < +\infty. \tag{2.4}$$

Assume that (2.4) does not hold. It will be assumed that

$$\varlimsup_{r\to\infty} (\log r)^{-1}\log f(ir) = +\infty,$$

since the case

$$\varlimsup_{r\to\infty} (\log r)^{-1}\log f(-ir) = +\infty$$

is treated analogously. By property b_1), there is an $r_0 > 0$ such that $f(ir)$ is positive and monotonically increasing for $r > r_0$.

Consider the rectangle

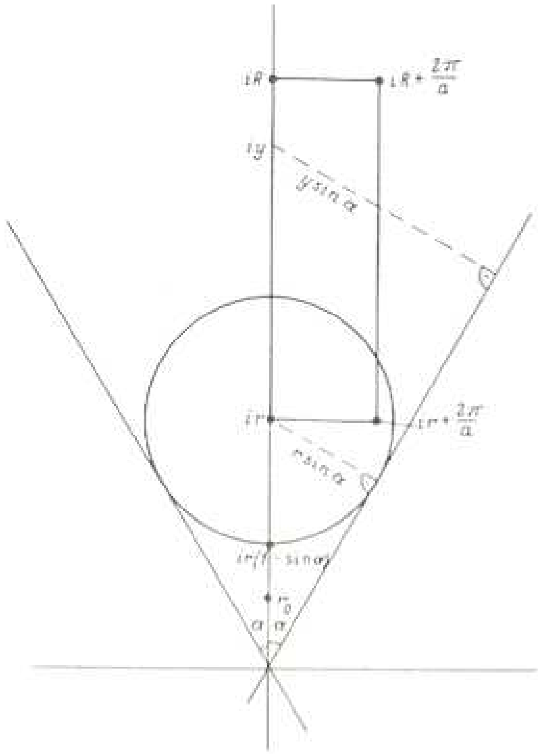
$$\{z: r \le \operatorname{Im} z \le R,\ 0 \le \operatorname{Re} z \le 2\pi/a\}$$

(see the figure), where the parameters are subject to the conditions

$$r_0/(1-\sin\alpha) < r < R < \infty, \qquad 2\pi/a = \tfrac{1}{2} r\sin\alpha. \tag{2.5}$$

This rectangle is contained in G_α, therefore f is analytic in it. Denoting by Π the rectangle's boundary, traversed clockwise, we have by the Cauchy theorem that

$$\int_\Pi e^{iaz} f(z)\,dz = 0, \qquad \int_\Pi e^{3iaz/2} f(z)\,dz = 0.$$



Separating out the imaginary parts, we get

$$\int_r^R \{f(iy) - \operatorname{Re} f(iy + 2\pi/a)\} e^{-ay}\, dy$$
$$= e^{-ar} \operatorname{Im} \int_0^{2\pi/2} e^{iax} f(x+ir)\, dx - e^{-aR} \operatorname{Im} \int_0^{2\pi/a} e^{iax} f(x+iR)\, dx, \quad (2.6)$$

$$\int_r^R \{f(iy) + \operatorname{Re} f(iy + 2\pi/a)\} e^{-3ay/2}\, dy$$
$$= e^{-3ar/2} \operatorname{Im} \int_0^{2\pi/a} e^{3iax/2} f(x+ir)\, dx$$
$$- e^{-3aR/2} \operatorname{Im} \int_0^{2\pi/a} e^{3iax/2} f(x+iR)\, dx. \quad (2.7)$$

By the inequality (2.2), the integrand on the left-hand side of (2.6) is non-negative. Therefore, it can only decrease if in it e^{-ay} is replaced by $e^{-3ay/2}$. Denote by (2.6 bis) the inequality obtained from (2.6) after this change. Adding (2.6 bis) to (2.7) and then performing simple estimates on the right-hand side, we have

$$\int_r^R f(iy) e^{-3ay/2}\, dy \le e^{-ar} \int_0^{2\pi/a} |f(x+ir)|\, dx + e^{-aR} \int_0^{2\pi/a} |f(x+iR)|\, dx. \quad (2.8)$$

To estimate the right-hand side of (2.8) we show that for $y > r_0/(1-\sin\alpha)$

$$\max\{|f(x+iy)|,\ 0 \le x \le \tfrac{1}{2}y\sin\alpha\} \le 2f(2iy). \tag{2.9}$$

Consider the function $d(\zeta) = f(\zeta y\sin\alpha + iy)$ on the disk $\{\zeta\colon |\zeta| < 1\}$. As ζ runs through this disk, the point $z = \zeta y\sin\alpha + iy$ runs through the disk $\{z\colon |z - iy| < y\sin\alpha\}$ in G_α. Therefore, g is analytic for $|\zeta| < 1$. By the properties of f, $\operatorname{Re} g(\zeta) \le \max\{f(\zeta y\sin\alpha i + iy)\colon -1 \le \zeta \le 1\} = f(iy(1+\sin\alpha)) < f(2iy)$. Using the Carathéodory inequality (2.1), we get

$$|f(\zeta y\sin\alpha + iy)| \le \frac{2|\zeta|}{1-|\zeta|}\{f(2iy) - f(iy)\} + f(iy).$$

Letting ζ run through the interval $0 \le \zeta \le \frac{1}{2}$, we arrive at (2.9).

Using (2.9) and recalling that $2\pi/a = \frac{1}{2}r\sin\alpha < \frac{1}{2}R\sin\alpha$ in view of (2.5), we get from (2.8) that

$$\int_r^R \{f(iy)e^{-3ay/2}\,dy \le \frac{4\pi}{a}\{e^{-ar}f(2ir) + e^{-aR}f(2iR)\}. \tag{2.10}$$

It follows from condition (2.3) that there exists a sequence of values $R \uparrow +\infty$ along which $\log f(2iR) = o(R)$. Letting R go to ∞ along this sequence, we get from (2.10) that

$$\int_r^\infty f(iy)e^{-3ay/2}\,dy \le \frac{4\pi}{a}e^{-ar}f(2ir). \tag{2.11}$$

Since $f(iy)$ is increasing and positive,

$$\int_r^\infty f(iy)e^{-3ay/2}\,dy \ge f(4ir)\int_{4r}^\infty e^{-3ay/2}\,dy = \frac{2}{3a}e^{-6ar}f(4ir),$$

therefore, by (2.11),

$$f(4ir) \le 6\pi e^{5ar}f(2ir).$$

The variables r and a are subject to the conditions (2.5), hence

$$f(4ir) \le 6\pi e^{20\pi/\sin\alpha}f(2ir), \qquad r \ge r_0/(1-\sin\alpha). \tag{2.12}$$

It is easy to see that if $u(x)$ is a function nondecreasing and positive on $[x_0, \infty)$ that satisfies the condition $u(2x) = O(u(x))$, $x \to \infty$, then $\log u(x) = O(\log x)$, $x \to \infty$. Therefore, (2.4) follows from (2.12).

§3. Generalization

A function φ analytic on the half-plane $\mathbb{C}_+ = \{z\colon \operatorname{Im} z > 0\}$ and continuous in $\mathbb{C}_+ \cup \mathbb{R}$ is said to be an *analytic characteristic function* (ACF) if its restriction to $\mathbb{R}$ is a characteristic function, i.e., (1.1) holds for $z \in \mathbb{R}$. It can be shown ([1], proof of Theorem 2.2.3) that if φ is an ACF, then for all $z \in \mathbb{C}_-$ the integral in (1.1) converges absolutely, and the representation (1.1) remains in force. Corresponding to ACF's are probability measures that are sufficiently rapidly decreasing at minus infinity. Since ACF's are representable in $\mathbb{C}_-$ in the form (1.1), they have the ridge property in $\mathbb{C}_-$. Functions φ analytic in $\mathbb{C}_+$ and continuous in $\mathbb{C}_+ \cup \mathbb{R}$ with $\varphi(0) = 1$ that have this property are called *analytic ridge functions* (ARF's); they form a broader class ([1], Russian pp. 46–47, English pp. 31–32) than the class of ACF's. By analogy with the order of an entire function, the order of an ARF φ is defined to be the number

$$\rho_+[\varphi] = \overline{\lim_{r\to\infty}}(\log r)^{-1}\log\log M_+(r,\varphi)(M_+(r,\varphi)$$
$$= \max\{|\varphi(z)|: |z|\le r, \ \operatorname{Im} z \ge 0\}).$$

An analogue of the Marcinkiewicz theorem for ARF's was obtained in [14].

THEOREM D [14]. *If an ARF φ of finite order does not have zeros in $\mathbb{C}_+$, then $\rho_+[\varphi] \le 3$.*

The simple example $\varphi(z) = \exp\{iz^3\}$ (this entire function has the ridge property only on $\mathbb{C}_+$, therefore, it is an ARF, but not an ERF!) proves that the estimate for $\rho_+[\varphi]$ is sharp. As shown in [14], this estimate is sharp also in the class of ACF's.

Theorem D can obviously be regarded also as a generalization of the result in [10] cited above. An analogue of Theorem C was obtained in [11] for ARF's:

THEOREM C′ [11]. *Suppose that an ARF φ of finite order does not have zeros in the domain $G_\alpha \cap \mathbb{C}$, $0 < \alpha < \pi/2$. Then $\rho_+[\varphi] < \max(\pi/\alpha, 4)$.*

It was established in [12] that the condition of finiteness of order in Theorem D can be replaced by the conditions

$$\underline{\lim_{r\to\infty}} r^{-1}\log\log M_+(r,\varphi) = 0. \tag{3.1}$$

The question arises as to whether the same change can be made in Theorem C′.

It is easy to see that any ARF φ has properties analogous to properties a)–c) of an ERF used in the proof of Theorem 1, namely: a″) $\varphi(iy) > 0$, $y \in \mathbb{R}_+$; b″) the function $\varphi(iy)$ is convex on $\mathbb{R}_+$; and c″) $M_+(r,\varphi) = \max\{\varphi(ir), 1\}$, $r \in \mathbb{R}_+$. If an ARF φ does not have zeros in $G_\alpha \cap \mathbb{C}_+$ and satisfies condition (3.1), then, denoting by f the analytic branch of $\log\varphi$ distinguished by the condition $f(0) = 0$ on $G_\alpha \cap \mathbb{C}_+$, we can repeat the arguments carried out in the proof of Theorem 1, and conclude that $\rho_+[\varphi] < \infty$. This and Theorem C′ give us the following fact.

THEOREM 3. *Suppose that an ARF φ satisfies condition* (3.1) *and does not have zeros in $G_\alpha \cap \mathbb{C}_+$. Then $\rho_+[\varphi] < \max(\pi/\alpha, 4)$.*

REMARK 2. In the paper of Vishnyakova mentioned in Remark 1 a sharp estimate for $\rho_+[\varphi]$ is found under the conditions of Theorem C′. This estimate has the form $\rho_+[\varphi] \le \gamma_1(\alpha)$, where $\gamma_1(\alpha)$ is defined for $0 < \alpha \le \pi/6$ just as $\gamma(\alpha)$ is defined in Remark 1, and $\gamma_1(\alpha) = 3$ for $\pi/6 \le \alpha \le \pi/2$. Obviously, it follows from this and Theorem 3 that the estimate $\rho_+[\varphi] \le \gamma_1(\alpha)$ remains in force also under the conditions of Theorem 3

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Physico-Technical Institute of Low Temperatures
Academy of Sciences of the Ukrainian SSR
Khar′kov

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Translated by H. H. McFADEN