On the growth of ridge functions non-vanishing in an angular domain

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Abstract

For an entire ridge function of finite order ρ which is non-vanishing in the angle $\{z : |\arg z - \pi/2| < \alpha\} \cup \{z : |\arg z + \pi/2| < \alpha\}$, $0 < \alpha \le \pi/2$, the sharp estimate of ρ in terms of α is obtained. Analogous result is obtained for ridge functions analytic in the upper half-plane.

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An entire function f, $f(0) = 1$, is called a ridge function if it satisfies the inequality

$$|f(z)| \le f(i\mathrm{Im} z). \quad (1)$$

If f is analytic in the upper half-plane $\boldsymbol{C}_+$ and satisfies (1) there, then we call it an analytic in $\boldsymbol{C}_+$ ridge function. In particular, all entire or analytic in $\boldsymbol{C}_+$ characteristic functions of probability distributions are ridge functions.

The well-known Marcinkiewicz theorem [1] states that, if entire ridge function of finite order ρ has a few zeros in some sense, then $\rho \le 2$. This theorem has been strenghtened and generalized in many directions (see the bibliography in [2]). In particular, I. P. Kamynin [3] proved that, if an analytic in $\boldsymbol{C}_+$ ridge function of a finite order ρ has no zeros at all, then $\rho \le 3$. The examples of the entire characteristic function of the Gauss distribution $f(z) = \exp(-\gamma z^2 + i\beta z)$, ($\gamma > 0$, $\beta \in \boldsymbol{R}$), and the analytic in $\boldsymbol{C}_+$ characteristic function $f(z) = (1 - iz)^{-1} \exp(iz^3 - 3z^2)$ (constructed by I. P. Kamynin) show that both estimates are the best possible. In [4] the question about estimate of the order of an entire or analytic in $\boldsymbol{C}_+$ ridge function non-vanishing in some angle is considered. The following two theorems are proved.

Theorem A ([4]). If an analytic in $\boldsymbol{C}_+$ ridge function of finite order ρ does not vanish in the angle $\{z : |\arg z - \pi/2| < \alpha\}$, $0 < \alpha \le \pi/2$, then $\rho < \max(4; \pi/\alpha)$.

Theorem B ([4]). If an entire ridge function of finite order ρ does not vanish in the angle $\{z : |\arg z - \pi/2| < \alpha\} \cup \{z : |\arg z + \pi/2| < \alpha\}$, $0 < \alpha \le \pi/2$, then

$$\rho \begin{cases} \le 2, & \pi/4 \le \alpha \le \pi/2; \\ < \pi/\alpha, & 0 < \alpha < \pi/4. \end{cases} \tag{2}$$

We show that the estimates for ρ given in Theorem A and Theorem B can be improved and obtain the best possible estimates.

Denote by $\gamma(\alpha)$, $0 < \alpha \le \pi/6$, the solution of the equation

$$\cos^{\gamma}(\alpha + \pi/\gamma) = -\cos(\gamma\alpha) \tag{3}$$

which belongs to the interval $(\pi/(2\alpha), \pi/\alpha)$. It is easy to see that the function $\gamma(\alpha)$ decreases and the following asymptotic equality holds

$$\gamma(\alpha) = \frac{\pi}{\alpha} - 2\sqrt{\frac{\pi}{\alpha}}(1 + o(1)), \quad \alpha \to 0. \tag{4}$$

Theorem 1. Let f be an analytic in $\boldsymbol{C}_+$ ridge function of finite order ρ non-vanishing in the angle $\{z : |\arg z - \pi/2| < \alpha\}$, $0 < \alpha \le \pi/2$. Then

$$\rho \le \begin{cases} \gamma(\alpha), & 0 < \alpha \le \pi/6; \\ 3 = \gamma(\pi/6), & \pi/6 < \alpha \le \pi/2. \end{cases} \tag{5}$$

Theorem 2. Let f be an entire ridge function of finite order ρ non-vanishing in the angle $\{z : |\arg z - \pi/2| < \alpha\} \cup \{z : |\arg z + \pi/2| < \alpha\}$, $0 < \alpha \le \pi/2$. Then

$$\rho \le \begin{cases} \gamma(\alpha), & 0 < \alpha \le \pi/6; \\ \pi/(2\alpha), & \pi/6 < \alpha \le \pi/4; \\ 2, & \pi/4 < \alpha \le \pi/2. \end{cases} \tag{6}$$

Remark. In [5], in particular, the following theorem has been proved.

Theorem. If an analytic in $\boldsymbol{C}_+$ ridge function f does not vanish in the angle $\{z : |\arg z - \pi/2| < \beta\}$ for some $\beta \in (0; \pi/2]$ and

$$\lim_{y} \inf_{\to +\infty} y^{-1} \log^+ \log^+ |f(iy)| = 0,$$

then the function f is of finite order.

Using this theorem, it is possible to weaken the condition of finiteness of the order in theorems 1 and 2.

Proof of Theorem 1. It is sufficient to prove the theorem for $\alpha \in (0; \pi/6)$. Assume that Theorem 1 is not valid. Let f be an analytic in $\boldsymbol{C}_+$ ridge function which does not vanish in the angle $\{z : |\arg z - \pi/2| < \alpha\}$, $0 < \alpha < \pi/6$, and has the finite order $\rho > \gamma(\alpha) \geq \gamma(\pi/6) = 3$. Let $a_k = r_k e^{i\varphi_k}$ be zeros of $f(iz)$. We shall use the following notations:

$$u(z) = \log |f(iz)|; \tag{7}$$

$$v_R(z) = \operatorname{Im}(e^{i\rho\alpha} z^{-\rho} + z^\rho e^{-i\rho\alpha} R^{-2\rho}) = (|z|^{-\rho} - |z|^\rho R^{-2\rho}) \sin \rho(\alpha - \arg z); \tag{8}$$

$$\prod\nolimits_R = \{z : 1 < |z| < R, 0 < \arg z < \alpha + \pi/\rho\} \quad (R > 1); \tag{9}$$

$$\beta = \alpha + \pi/\rho \quad (0 < \beta < \pi/2). \tag{10}$$

Let us apply the Green formula in the domain $\prod_R$ to the pair $(u(z), v_R(z))$. Since $v_R(z)$ is harmonic in $\prod_R$ for every $R > 1$ and, moreover, from (1) follows $\partial u/\partial y|_{(x,0)} = 0$, $1 \leq x \leq R$, we obtain

$$\int_1^R \Big\{(-\cos \rho\alpha) u(x) - u(xe^{i\beta})\Big\}(x^{-\rho-1} - x^{\rho-1} R^{-2\rho}) dx = 2R^{-\rho} \int_0^\beta u(Re^{i\theta}) \sin \rho(\alpha - \theta) d\theta + 2\pi\rho^{-1} \sum_{a_k \in \prod_R} (r_k^{-\rho} - r_k^\rho R^{-2\rho}) \sin \rho(\alpha - \varphi_k) + C_1 + C_2 R^{-2\rho}, \tag{11}$$

where C_1 and C_2 are constants not depending on R.

Denote

$$A(R) = \int_1^R \Big\{(-\cos \rho\alpha) u(x) - u(xe^{i\beta})\Big\}(x^{-\rho-1} - x^{\rho-1} R^{-2\rho}) dx; \tag{12}$$

$$B(R) = 2R^{-\rho}\int_0^{\beta} u(Re^{i\theta})\sin\rho(\alpha-\theta)d\theta; \qquad (13)$$

$$S(R) = 2\pi\rho^{-1}\sum_{a_k\in\prod_R}(r_k^{-\rho} - r_k^{\rho}R^{-2\rho})\sin\rho(\alpha-\varphi_k). \qquad (14)$$

Using these notations we can rewrite formula (11) in such a way

$$A(R) = B(R) + S(R) + C_1 + C_2R^{-2\rho}. \qquad (15)$$

Subtracting from (15) the formula obtained by replacing R by r in (15), $1 < r < R < \infty$, we obtain

$$A(R) - A(r) = B(R) - B(r) + S(R) - S(r) + C_2(R^{-2\rho} - r^{-2\rho}). \qquad (16)$$

Now let us estimate from below the left-hand side of (16). Since f is a ridge function, $u(x)$ is convex on R ([1]). Therefore without loss of generality we may asuume that $u(x)$ is positive and monotonically increases in x when $x > 0$. Further we shall denote by K positive constants not depending on r, R not necessary equal. We have

$$\begin{aligned}
A(R) - A(r) = &\int_1^R \Big(u(x) - u(xe^{i\beta})\Big)(x^{-\rho-1} - x^{\rho-1}R^{-2\rho})dx - \\
&\int_1^r \Big(u(x) - u(xe^{i\beta})\Big)(x^{-\rho-1} - x^{\rho-1}r^{-2\rho})dx - \\
&(1+\cos\rho\alpha)\int_1^R u(x)(x^{-\rho-1} - x^{\rho-1}R^{-2\rho})dx + \\
&(1+\cos\rho\alpha)\int_1^r u(x)(x^{-\rho-1} - x^{\rho-1}r^{-2\rho})dx \geq \\
&\int_r^R \Big(u(x) - u(xe^{i\beta})\Big)(x^{-\rho-1} - x^{\rho-1}R^{-2\rho})dx - \\
(1+\cos\rho\alpha)\int_r^R u(x)x^{-\rho-1}dx - &(1+\cos\rho\alpha)\int_1^r u(x)x^{\rho-1}r^{-2\rho}dx \geq \\
&\int_r^R \Big(u(x) - u(x\cos\beta)\Big)(x^{-\rho-1} - x^{\rho-1}R^{-2\rho})dx - \\
(1+\cos\rho\alpha)\int_r^R u(x)x^{-\rho-1}dx - &(1+\cos\rho\alpha)\int_1^r u(x)x^{\rho-1}r^{-2\rho}dx \geq \\
&\int_r^R \Big\{(-\cos\rho\alpha)u(x) - u(x\cos\beta)\Big\}x^{-\rho-1}dx - \\
\int_r^R u(x)x^{\rho-1}R^{-2\rho}dx - &(1+\cos\rho\alpha)\int_1^r u(x)x^{\rho-1}r^{-2\rho}dx \geq
\end{aligned}$$

$$(-\cos\rho\alpha)\int_r^R u(x)x^{-\rho-1}dx - \cos^\rho\beta\int_{r\cos\beta}^{R\cos\beta} u(x)x^{-\rho-1}dx -$$

$$Ku(R)R^{-\rho} - Ku(r)r^{-\rho}(-\cos\rho\alpha - \cos^\rho\beta)\int_r^R u(x)x^{-\rho-1}dx -$$

$$\cos^\rho\beta\int_{r\cos\beta}^{r} u(x)x^{-\rho-1}dx - Ku(R)R^{-\rho} - Ku(r)r^{-\rho} \geq$$

$$(-\cos\rho\alpha - \cos^\rho\beta)\int_r^R u(x)x^{-\rho-1}dx - Ku(R)R^{-\rho} - Ku(r)r^{-\rho}. \qquad (17)$$

The function $h_\alpha(\rho) = -\cos\rho\alpha - \cos^\rho\beta = -\cos\rho\alpha - \cos^\rho(\alpha + \pi/\rho)$ has the unique root $\gamma(\alpha)$ on the interval $[\pi/(2\alpha); \pi/\alpha)$, and $h_\alpha(\pi/\alpha) > 0$. Therefore using our assumption $\rho > \gamma(\alpha)$ we have $\varepsilon = h_\alpha(\rho) > 0$. Substituting estimate (17) into (16) and dividing by $\varepsilon > 0$ we obtain

$$\int_r^R u(x)x^{-\rho-1}dx \leq K\Big(B(R) - B(r)\Big) + K\Big(S(R) - S(r)\Big) +$$
$$Ku(R)R^{-\rho} + Ku(r)r^{-\rho}. \qquad (18)$$

(we have applied positivity and monotonic increasing of $u(x)$).

Let us estimate the right-hand side of (18) from above.

1. To estimate $B(R)$ we use the following Lemma.

Lemma 1. ([4])

$$|B(R)| \leq Ku(R)R^{-\rho}. \qquad (19)$$

Proof of Lemma 1. We use the Carleman formula for function $f(z)$ ([8, p.224]):

$$\frac{1}{\pi R}\int_0^\pi \log|f(Re^{i\theta})|\sin\theta d\theta + \frac{1}{2}\int_1^R (t^{-2} - R^{-2})\log|f(t)f(-t)|dt =$$
$$\sum_{1<r_k<R}(r_k^{-1} - r_k R^{-2})\cos\varphi_k + b_1 + b_2R^{-2}, \qquad (20)$$

where $1 \leq R < \infty$; b_1, b_2 do not depend on R. Since $f(z)$ is a ridge function, $f(0) = 1$, we have $|f(t)| \leq 1$, $t \in \boldsymbol{R}$. Hence (20) implies

$$\frac{1}{\pi R}\int_0^\pi \log|f(Re^{i\theta})|\sin\theta d\theta \geq b_1 + b_2R^{-2}. \qquad (21)$$

Using the fact that $0 < \beta < \pi/2$ and (21), we obtain

$$|B(R)| \leq KR^{-\rho}\int_{\pi/2}^{\pi/2+\beta}\Big|\log|f(Re^{i\theta})|\Big|d\theta \leq$$

$$KR^{-\rho}\int_{\pi/2}^{\pi/2+\beta}\Big(\log^+|f(Re^{i\theta})|+\log^-|f(Re^{i\theta})|\Big)\sin\theta d\theta \le$$

$$KR^{-\rho}u(R)+KR^{-\rho}\int_0^{\pi}\log^-|f(Re^{i\theta})|\sin\theta d\theta \le$$

$$KR^{-\rho}\Big(u(R)+\pi b_1 R+\pi b_2 R^{-1}\Big). \qquad (22)$$

Since f is an unbounded in $\boldsymbol{C}_+$ analytic ridge function we have by [1] $R = O(\log^+ f(iR))$, $R\to+\infty$. Therefore from (22) we obtain the statement of Lemma 1. ■

Using the statement of Lemma 1 we obtain

$$B(R)-B(r)\le Ku(R)R^{-\rho}+Ku(r)r^{-\rho}. \qquad (23)$$

2. To estimate $S(R)-S(r)$ we shall use the fact that function $f(iz)$ does not vanish in the angle $\{z : 0<\arg z<\alpha\}$. We have

$$S(R)-S(r)=2\pi\rho^{-1}\Bigg(\sum_{\substack{1<r_k<R\\ \alpha<\varphi_k<\beta}}(r_k^{-\rho}-r_k^{\rho}R^{-2\rho})\sin\rho(\alpha-\varphi_k)-$$

$$\sum_{\substack{1<r_k<r\\ \alpha<\varphi_k<\beta}}(r_k^{-\rho}-r_k^{\rho}r^{-2\rho})\sin\rho(\alpha-\varphi_k)\Bigg)=$$

$$2\pi\rho^{-1}\Bigg(\sum_{\substack{1<r_k<r\\ \alpha<\varphi_k<\beta}}r_k^{\rho}(r^{-2\rho}-R^{-2\rho})\sin\rho(\alpha-\varphi_k)+$$

$$\sum_{\substack{r<r_k<R\\ \alpha<\varphi_k<\beta}}(r_k^{-\rho}-r_k^{\rho}R^{-2\rho})\sin\rho(\alpha-\varphi_k)\Bigg)\le 0. \qquad (24)$$

Substituting (23) and (24) into (18), we obtain

$$\int_r^R u(x)x^{-\rho-1}dx\le Ku(R)R^{-\rho}+Ku(r)r^{-\rho}. \qquad (25)$$

To complete the proof we shall use the following elementary lemma:

Lemma 2 ([4]). Let $w(x)\ge 0$ be continuously differentiable non-decreasing function on $[1;\infty)$ and for all r and R, $r<R$, the inequality

$$w(R)-w(r)\le KRw'(R)+Krw'(r). \qquad (26)$$

holds. Then for all sufficiently large x we have:

1) if $w(\infty) = \infty$, then $w(x) \geq Kx^{\delta}$;

2) if $w(\infty) < \infty$, then $w(\infty) - w(x) \leq Kx^{-\delta}$,

where δ is a positive number.

Proof of Lemma 2.

1. Let $w(\infty) = \infty$. Then by (26) we have $Rw'(R) \to \infty$, $R \to \infty$. Substituting $r = 1$ into (26) and using the last statement, we have

$$w(R) \leq KRw'(R) \tag{27}$$

(with a larger K).

Hence

$$\frac{w'(R)}{w(R)} \geq \frac{1}{KR}. \tag{28}$$

Integrating the last inequality from 1 to r, we obtain statement 1) of Lemma2.

2. Let $w(\infty) < \infty$. Then there exists a sequence $R_k \to \infty$ such that $R_k w'(R_k) \to 0$, $k \to \infty$. Making $R \to \infty$ along the sequence $\{R_k\}_{k=1}^{\infty}$ we have

$$w(\infty) - w(r) \leq Krw'(r), \tag{29}$$

hence

$$\frac{w'(r)}{w(\infty) - w(r)} \geq \frac{1}{Kr} \tag{30}$$

Integrating last inequality from 1 to r, we obtain

$$-\log \frac{w(\infty) - w(r)}{w(\infty) - w(1)} \geq \log r^{1/K}, \tag{31}$$

that implies statement 2) of Lemma 2. ■

Using Lemma 2 with $w(x) = \int_1^x u(t)t^{-\rho-1}dt$, we obtain contradiction to the assumption that ρ is the order of the function f.

Theorem 1 is proved. ■ ■

Proof of the Theorem 2. Since f is an analytic in $\boldsymbol{C}_+$ ridge function we have $\rho \leq \gamma(\alpha)$ for $0 < \alpha \leq \pi/6$. Therefore it is sufficient to prove Theorem 2 for $\pi/6 < \alpha < \pi/4$, $2 < \rho \leq 3$. Without loss of generality we can assume that f is an even function (we can consider the function $f(z)f(-z)$ instead of $f(z)$). Suppose the contrary that there exists an even entire ridge function f non-vanishing in the angle $\{z : |\arg z - \pi/2| < \alpha\} \cup \{z : |\arg z + \pi/2| < \alpha\}$

and having the finite order $\rho > \pi/(2\alpha)$. Let us denote by $a_k = r_k e^{i\varphi_k}$ zeros of function $f(iz)$ lying in the half-plane $\{z : \mathrm{Re} z > 0\}$. We use the notations $u(z)$ and $v_R(z)$ introduced in the proof of the Theorem 1. Denote $C_R = \{z : 1 < |z| < R,\ 0 < \arg z < \pi/2\}$, $R > 1$. Let us use the Green formula in C_R for functions u and v_R. Since v_R is harmonic in C_R, and $f(z)$ is even and satisfies (1), we obtain $\partial u(x,0)/\partial y = 0$, $\partial u(0,y)/\partial x = 0$. Hence

$$\int_1^R \Big\{(-\cos\rho\alpha)u(x) - (-\cos\rho(\alpha-\pi/2))u(ix)\Big\}(x^{-\rho-1} - x^{\rho-1}R^{-2\rho})dx = 2R^{-\rho}\int_0^{\pi/2} u(Re^{i\theta})\sin\rho(\alpha-\theta)d\theta + 2\pi\rho^{-1}\sum_{a_k\in C_R}(r_k^{-\rho} - r_k^{\rho}R^{-2\rho})\sin\rho(\alpha-\varphi_k) + C_1 + C_2R^{-2\rho}, \quad (32)$$

where C_1 and C_2 are positive constants. By our assumption $(-\cos\rho(\alpha - \pi/2)) > 0$ holds. Since f, $f(0) = 1$ is an even entire ridge function, the function $u(x)$ is positive and monotonically increases in x, $x > 0$. As in the proof of Theorem 1 we denote

$$A(R) = \int_1^R \Big\{(-\cos\rho\alpha)u(x) - (-\cos\rho(\alpha-\pi/2))u(ix)\Big\} \times (x^{-\rho-1} - x^{\rho-1}R^{-2\rho})dx; \quad (33)$$

$$B(R) = 2R^{-\rho}\int_0^{\pi/2} u(Re^{i\theta})\sin\rho(\alpha-\theta)d\theta; \quad (34)$$

$$S(R) = 2\pi\rho^{-1}\sum_{a_k\in C_R}(r_k^{-\rho} - r_k^{\rho}R^{-2\rho})\sin\rho(\alpha-\varphi_k). \quad (35)$$

Substacting from (32) the formula obtained from (32) by changing R by r, $1 < r < R < \infty$, we have

$$A(R) - A(r) = B(R) - B(r) + S(R) - S(r) + C_2(R^{-2\rho} - r^{-2\rho}). \quad (36)$$

Let us estimate the left-hand part of (36) from below

$$A(R) - A(r) = \int_1^R (-\cos\rho\alpha)u(x)(x^{-\rho-1} - x^{\rho-1}R^{-2\rho})dx - \int_1^r (-\cos\rho\alpha)u(x)(x^{-\rho-1} - x^{\rho-1}r^{-2\rho})dx - (-\cos\rho(\alpha-\pi/2))\int_1^R u(ix)(x^{-\rho-1} - x^{\rho-1}R^{-2\rho})dx +$$

$$
\begin{aligned}
(-\cos\rho(\alpha-\pi/2))\int_1^r u(ix)(x^{-\rho-1}-x^{\rho-1}r^{-2\rho})dx \geq \\
x(-\cos\rho\alpha)\int_1^R u(x)(x^{-\rho-1}-x^{\rho-1}R^{-2\rho})dx - \\
(-\cos\rho\alpha)\int_1^r u(x)(x^{-\rho-1}-x^{\rho-1}r^{-2\rho})dx \geq \\
(-\cos\rho\alpha)\int_r^R u(x)(x^{-\rho-1}-x^{\rho-1}R^{-2\rho})dx \geq \\
(-\cos\rho\alpha)\int_r^R u(x)x^{-\rho-1}dx - Ku(R)R^{-\rho}. \qquad (37)
\end{aligned}
$$

The upper estimate of $B(R)-B(r)$ we obtain in the same way as in the proof of Theorem 1. We have

$$B(R)-B(r) \leq Ku(R)R^{-\rho} + Ku(r)r^{-\rho}. \qquad (38)$$

Since function f has no zeros in the angle $\{z : |\arg z - \pi/2| < \alpha\}$

$$S(R)-S(r) \leq 0 \qquad (39)$$

holds.

Substituting (37), (38), and (39) into (36), we obtain

$$\int_r^R u(x)x^{-\rho-1}dx \leq Ku(R)R^{-\rho} + Ku(r)r^{-\rho}. \qquad (40)$$

As in the proof of Theorem 1 we see that (40) contradicts to the fact that ρ is the order of function f.

Theorem 2 is proved. ■ ■

The next statement shows the sharpness of the estimates of ρ in Theorems 1 and 2.

Theorem 3. 1. For each α, $0 < \alpha \leq \pi/2$, there exists an entire characteristic function f nonvanishing in the angle $\{z : |\arg z - \pi/2| < \alpha\} \cup \{z : |\arg z + \pi/2| < \alpha\}$ and having the order

$$\rho = \begin{cases} \gamma(\alpha), & 0 < \alpha \leq \pi/6; \\ \pi/(2\alpha), & \pi/6 < \alpha \leq \pi/4; \\ 2, & \pi/4 < \alpha \leq \pi/2. \end{cases} \qquad (41)$$

2. For each α, $0 < \alpha \leq \pi/2$, there exists an analytic in $\boldsymbol{C}_+$ characteristic function f non-vanishing in the angle $\{z : |\arg z - \pi/2| < \alpha\}$ and having the order

$$\rho = \begin{cases} \gamma(\alpha), & 0 < \alpha \leq \pi/6; \\ 3, & \pi/6 < \alpha \leq \pi/2. \end{cases} \tag{42}$$

Proof of Theorem 3. **1.** For $\alpha \in [\pi/4; \pi/2]$ we can take the Gauss characteristic function $f(z) = \exp\{-\gamma z^2 + i\beta z\}$ ($\gamma > 0$; $\beta \in \boldsymbol{R}$) as an example. Consider $\alpha \in (0; \pi/4)$. For constructing an example in this case we shall use a result of [6]. Let $h(\theta)$ be a 2π-periodic function on $\boldsymbol{R}$, let ρ be a number greater than 2. Suppose that the following conditions are satisfied:

1) $h(\theta)$ is a ρ-trigonometrically convex function;
2) $\exists \delta > 0, A > 0; h(\theta) = A\cos\rho(\pi/2 - \theta)$ for $|\pi/2 - \theta| < \delta$;
3) $h(\pi/2 + \theta) = h(\pi/2 - \theta)$, for $\theta \in [0; \pi/2]$;
4) $h(\pi/2 + \theta) \leq h(\pi/2)\cos^\rho(\pi/2 - \theta)$, for $\theta \in [0; \pi/2]$.

Theorem [6]. There exists an entire characteristic function f of order ρ having completely regular growth (in sense Levin-Pfluger) with the indicator $h(\theta)$ and non-vanishing inside the angles where $h(\theta)$ if ρ-trigonometric.

We shall construct function $h(\theta)$ satisfying conditions 1)–4) and such that $h(\theta)$ is ρ-trigonometric for $\theta \in [\pi/2 - \alpha; \pi/2 + \alpha]$. Since $h(\theta)$ will be an even function satisfying 3), it is sufficient to construct $h(\theta)$ when $\theta \in [\pi/2; \pi]$.

a). Consider $\alpha \in [0; \pi/6]$, $\rho = \gamma(\alpha) \geq 3$. Define $h(\theta)$ by the formula

$$h(\theta) = \begin{cases} \cos\rho(\pi/2 - \theta), & \theta \in [\pi/2; \pi/2 + \alpha]; \\ -\cos^{\rho-1}(\alpha + \pi/\rho) \times & \\ \cos\rho(\theta + \alpha/\rho + \pi/\rho^2 - \alpha - \pi/2), & \theta \in [\pi/2 + \alpha; \pi/2 + \alpha + \pi/\rho]; \\ \cos^\rho(\pi/2 - \theta), & \theta \in [\pi/2 + \alpha + \pi/\rho; \pi]. \end{cases} \tag{43}$$

It is easy to verify that $h(\theta)$ is a ρ-trigonometrically convex function (we use the fact that $\rho = \gamma(\alpha)$ satisfies equation $\cos^\rho(\alpha + \pi/\rho) = -\cos\rho\alpha$) and $h(\theta)$ satisfies 1)–4). Therefore Theorem of [6] cited above yields that there exists an entire characteristic function f of order ρ and of completely regular growth having indicator $h(\theta)$. Since $h(\theta)$ is ρ-trigonometric for $\theta \in [-\pi/2 - \alpha; -\pi/2 + \alpha] \cup [\pi/2 - \alpha; \pi/2 + \alpha]$, the function f does not vanish in the angle $\{z : |\arg z - \pi/2| < \alpha\} \cup \{z : |\arg z + \pi/2| < \alpha\}$.

2. Consider $\alpha \in [\pi/6; \pi/4], \rho = \pi/(2\alpha)$. Define

$$h(\theta) = \begin{cases} \cos\rho(\pi/2 - \theta), & \theta \in [\pi/2; \pi/2 + \alpha]; \\ 0, & \theta \in [\pi/2 + \alpha; \pi]. \end{cases} \tag{44}$$

The function $h(\theta)$ is ρ-trigonometrically convex and satisfies 1)–4). Therefore the above Theorem of [6] yields that there exists an entire characteristic function f of order ρ and of completely regular growth having indicator $h(\theta)$. Since $h(\theta)$ is ρ-trigonometric for $\theta \in [-\pi/2-\alpha; -\pi/2+\alpha] \cup [\pi/2-\alpha; \pi/2+\alpha]$, the function f does not vanish in the angle $\{z : |\arg z - \pi/2| < \alpha\} \cup \{z : |\arg z + \pi/2| < \alpha\}$.

2. When $\alpha \in (0; \pi/6]$ we can take as example the entire function constructed in a). When $\alpha \in [\pi/6; \pi/2]$ we can take $f(z) = (1 - iz)^{-1} \exp(iz^3 - 3z^2)$.

Theorem 3 is proved. ■ ■

Remark. Using methods of the theory of the cluster sets of subharmonic functions developed by V. S. Azarin [9], A. E. Fryntov in [7] proved independently some more general statement than Theorem 2.

References

[1] Yu. V. Linnik and I. V. Ostrovskii. Decomposition of random variables and vectors, Amer. Math. Soc., Providence, R.I., 1977.

[2] I. V. Ostrovskii. The arithmetic of probability distributions, Theor. Probab. Appl., 1986, vol. 31, p. 1–24.

[3] I. P. Kamynin. A generalization of the Marcinkiewicz theorem on entire characteristic functions of probability distribution, Zap. Nauchn. Sem. LOMI, 1979, English transl.: J. Soviet Math., vol. 20, No. 3, 1982.

[4] A. M. Vishnyakova, I. V. Ostrovskii. An analogue of the Marcinkiewicz theorem for entire ridge functions non-vanishing in an angular domain, Dokl. Akad. Nauk Ukraine, Ser. A. 1987, No. 9, p. 8–11 (Russian).

[5] A. M. Vishnyakova, I. V. Ostrovskii, A. M. Ulanovskii. On a conjecture of Ju. V. Linnik, Algebra & Analiz, 1991, vol. 2, No. 4, p. 82–90, English transl.: Leningrad Math. J., 1991, vol. 2, No. 4, p. 765–773.

[6] A. A. Gol'dberg and I. V. Ostrovskii. Indikators of entire Hermitian-positive functions of finite order, Sibirsk. Mat. Zh., 1982, vol. 23, No. 6, p. 55–73, English transl.: Siberian Math. J., 1982, p. 804–820.

[7] A. E. Fryntov. About one property of cone generated by translations of subharmonic ridge function, Analiticheskie metody v teorii veroyatnostey i teorii operatorov, Kiev, "Naukova dumka", 1990, p. 33–39 (Russian).

[8] B. Ja. Levin. Distribution of Zeros of Entire Functions, AMS, Providence, R.I., 1980.

[9] V. S. Azarin. Cluster sets of entire and subharmonic functions. In "Itogi nauki i techniki, Ser. Sovrem. problemy mat.", VINITI, Moscow 1991, 52–68 (Russian).