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Fine structure of the Vavilov–Cherenkov radiation

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We found relativistic quantum corrections to the one-photon Cherenkov emission. It is proved that, in the absence of dispersion, the Vavilov–Cherenkov radiation fills the whole Cherenkov cone (in the Tamm–Frank theory the Vavilov–Cherenkov radiation for the fixed refractive index is confined to the surface of the Cherenkov cone). The radiation intensity reaches the maximum inside the Cherenkov cone. It turns out that photons with different energies fly at different angles in the interval from zero up to the Cherenkov angle corresponding to the initial charge velocity. The visible light region, where the Vavilov–Cherenkov radiation is usually observed, is surrounded by the low intensity infrared region and by the high intensity one corresponding to high energy photons. The ratio of the radiation intensity at the maximum lying in the Roentgen part of the radiation spectrum to the radiation intensity in its visible part is about 10^4. Taking into account the medium dispersion leads to the appearance of the striped-like radiation structure inside the Cherenkov cone. Experimental data indicating the existence of a non-zero radiation field inside this cone are discussed. In the past, non-relativistic quantum corrections to the radiation intensity were found by Ginzburg. Yet, he did not analyse their influence for large energy–momentum transfer.

Keywords: Vavilov–Cherenkov radiation; quantum correction terms

1. Introduction

Theoretical explanation of the blue light observed in Cherenkov (1934) experiments originates with Frank & Tamm (1937) who associated it with the radiation of a charge moving uniformly with a velocity greater than the light velocity in medium. The radiation intensity obtained by them in the absence of medium dispersion was confined to the surface of the so-called Cherenkov cone defined by

$$\cos\theta = \frac{1}{\beta_0 n}. \tag{1.1}$$

Here, $\beta_0 = v_0/c$ and n is the medium refractive index. In the presence of dispersion, n and $\cos\theta$ continuously vary with ω. Correspondingly, the

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Cherenkov radiation fills the continuous sequence of Cherenkov cones corresponding to different frequencies (in the frequency regions where $n>1$).

In their calculations, Tamm and Frank suggested that the charge velocity was constant, thus disregarding the recoil effects and assuming smallness of the photon energy with respect to the energy of the initial charge. Since then, it is usually believed that the Vavilov–Cherenkov (VC) radiation for the fixed refractive index lies on the surface of the Cherenkov cone.

In the note (Vavilov 1934) accompanying the Cherenkov (1934) paper, Vavilov suggested that the radiation observed in Cherenkov experiments was due to the electron deceleration. Now we know (see discussion in Afanasiev 2004) that Vavilov was at least partly right since electrons were completely stopped in Cherenkov experiments (their thorough discussion may be found in Cherenkov's Doctor of Science dissertation; Cherenkov 1944), thus exhibiting deceleration. In his note, Vavilov made the following important remark: 'Thus, weak visible radiation may arise, although the boundary of bremsstrahlung and its maximum should be somewhere in the Roentgen region. It follows from this that the energy distribution in the visible region should rise towards the violet part of the spectrum, and the blue–violet part of the spectrum should be especially intensive.'

Later Ginzburg (1940) evaluated the photon emission angle for an arbitrary energy loss of the initial charge. He also found the radiation intensity in the non-relativistic approximation and showed that corrections to the Tamm–Frank formula are negligible in the visible and ultraviolet parts of the radiation spectrum. Yet, he did not analyse the behaviour of the radiation intensities for large energy–momentum transfer.

In Afanasiev & Stepanovsky (2003), devoted to the kinematics of the two-photon Cherenkov effect, one-photon kinematics was considered for the pedagogical purpose. It was shown there that the photon energy changes from zero (when the photon flies at the Cherenkov angle) up to some maximal value (when the photon flies in the direction of the initial charge). However, no radiation intensities were evaluated there. The aim of this consideration is to analyse the behaviour of the radiation intensities for large energy–momentum transfer.

The plan of our exposition is as follows. In §2, we reproduce the kinematic relations presented in Afanasiev & Stepanovsky (2003) with their modification needed for the subsequent exposition. In §3, we obtain the radiation intensity for the electron moving with a velocity greater than the light velocity in medium. In the kinematically permissible region the radiation intensity (as a function of frequency) either grows or reaches the maximal value in the hard Roentgen region in exact agreement with Vavilov's prediction. The radiation intensity for charged particles with spin zero found in §4 vanishes at the boundaries of the kinematically permissible region reaching the maximum inside it. It was shown theoretically (Afanasiev *et al.* 1999*a*,*b*) and experimentally (Stevens *et al.* 2001; Wahlstrand & Merlin 2003) that the inclusion of the medium dispersion leads to the appearance of additional radiation intensity maxima (or striped-like structure according to Wahlstrand & Merlin 2003) in the angular distribution of the radiation. A rather poor agreement between calculated and observed radiation intensities was attributed in Wahlstrand & Merlin (2003) to the radiation damping. Equally, it can be associated with quantum correction terms which were not taken into account in Afanasiev *et al.* (1999*a*,*b*), Stevens *et al.* (2001) and Wahlstrand & Merlin (2003). The available experimental data indicating the existence of the VC radiation inside the Cherenkov cone are analysed. To separate contributions of the quantum correction terms and the medium dispersion effects we consider the hypothetical dispersion free medium (thus, disregarding the ω dependence of n). Furthermore, since we operate in the spectral representation, the main formulae are valid in those frequency regions where $n>1$.

To avoid confusion, it should be mentioned that this paper has the same title as Afanasiev *et al.* (2003) dealing with a fine structure of radiation arising from the charge motion in a finite space interval. The resulting configuration of shock waves included the Cherenkov shock wave, bremsstrahlung and shock waves arising from the charge overcoming the medium light velocity barrier. In contrast, the present paper deals with the unbounded motion of a charge moving in medium with the velocity greater than the light velocity in medium. It is our goal to investigate the fine structure of the arising VC radiation.

2. The kinematics of the one-photon Cherenkov effect

Let a point-like charge e having the rest mass m_0 move in medium of the refractive index n. It emits a photon with the frequency ω. The conservation of energy and momentum gives

$$m_0 c^2 \gamma_0 = m_0 c^2 \gamma + \hbar\omega, \quad m_0 \boldsymbol{v}_0 \gamma_0 = m_0 \boldsymbol{v}\gamma + \frac{\hbar\omega n}{c}\boldsymbol{e}_\gamma. \tag{2.1}$$

Here, $\hbar$ is the Plank constant, $\boldsymbol{v}_0$ and $\boldsymbol{v}$ are the charge velocities before and after emitting the γ quantum, $\gamma=1/\sqrt{1-\beta^2}$, $\gamma_0=1/\sqrt{1-\beta_0^2}$; $\boldsymbol{e}_\gamma$ and ω are the unit vector in the direction of emitted γ quanta and its frequency; n is the medium refractive index taken at the frequency ω. We rewrite (2.1) in the dimensionless form

$$\gamma_0 = \gamma + \epsilon, \quad \boldsymbol{\beta}_0\gamma_0 = \boldsymbol{\beta}\gamma + \epsilon n \boldsymbol{e}_\gamma. \tag{2.2}$$

Here, $\boldsymbol{\beta}=\boldsymbol{v}/c$, $\boldsymbol{\beta}_0=\boldsymbol{v}_0/c$, $\epsilon=\hbar\omega/m_0c^2$. Let $\boldsymbol{v}_0$ be directed along the z-axis. We project all vectors on this axis and two others perpendicular to it

$$\left.\begin{aligned} \boldsymbol{\beta}_0 = \beta_0 \boldsymbol{e}_z, \quad \boldsymbol{\beta} = \beta[\boldsymbol{e}_z \cos\theta + \sin\theta(\boldsymbol{e}_x\cos\phi + \boldsymbol{e}_y \sin\phi)], \\ \boldsymbol{e}_\gamma = \boldsymbol{e}_z\cos\theta_\gamma + \sin\theta_\gamma(\boldsymbol{e}_x\cos\phi_\gamma + \boldsymbol{e}_y\sin\phi_\gamma). \end{aligned}\right\} \tag{2.3}$$

Substituting (2.3) into (2.2), one obtains

$$\left.\begin{aligned} \gamma_0 = \gamma + \epsilon, \quad \beta_0\gamma_0 = \beta\gamma\cos\theta + n\epsilon\cos\theta_\gamma, \\ \beta\gamma\sin\theta\cos\phi + n\epsilon\sin\theta_\gamma\cos\phi_\gamma = 0, \quad \beta\gamma\sin\theta\sin\phi + n\epsilon\sin\theta_\gamma\sin\phi_\gamma = 0. \end{aligned}\right\} \tag{2.4}$$

From the last two equations one finds

$$\sin\theta\sin(\phi-\phi_\gamma) = 0, \quad \sin\theta_\gamma\sin(\phi-\phi_\gamma) = 0. \tag{2.5}$$

For $\sin(\phi-\phi_\gamma)\neq 0$, one finds that $\sin\theta=\sin\theta_\gamma=0$. It can be shown (Afanasiev & Stepanovsky 2003) that this is a particular case of the solution $\sin(\phi-\phi_\gamma)=0$ considered below.

Let $\sin(\phi-\phi_\gamma)=0$. There are no physical solutions of (2.4) if $\phi=\phi_\gamma$. It remains only $\phi=\phi_\gamma+\pi$. Then,

$$\gamma_0=\gamma+\epsilon,\quad \beta_0\gamma_0=\beta\gamma\cos\theta+n\epsilon\cos\theta_\gamma,\quad \beta\gamma\sin\theta=n\epsilon\sin\theta_\gamma. \tag{2.6}$$

These equations have the following solution:

$$\cos\theta_\gamma=\frac{1}{\beta_0 n}\left[1+\frac{\epsilon(n^2-1)}{2\gamma_0}\right],\quad \cos\theta=\frac{\beta^2\gamma^2+\beta_0^2\gamma_0^2-n^2(\gamma_0-\gamma)^2}{2\beta\gamma\beta_0\gamma_0}. \tag{2.7}$$

The conditions that the r.h.s. of these equations should be smaller than 1 and greater than -1 lead to the following conditions:

$$\frac{|2n-\beta_0(n^2+1)|}{n^2+1-2n\beta_0}<\beta<\beta_0,\quad 0<\epsilon<\epsilon_{\rm right}\equiv\frac{2\gamma_0(\beta_0 n-1)}{n^2-1}. \tag{2.8}$$

For $\epsilon=0$, one gets

$$\cos\theta_\gamma=\cos\theta_{\rm c}\equiv 1/\beta_0 n,\quad \cos\theta=1,\quad \beta=\beta_0,$$

that is, the low-energy photon flies at the Cherenkov angle $\theta_{\rm c}$, while the charge moves in the forward direction with the energy almost coinciding with the initial one. For $\epsilon=\epsilon_{\rm right}$,

$$\cos\theta_\gamma=1,\quad \beta=\frac{|2n-\beta_0(n^2+1)|}{n^2+1-2n\beta_0},\quad \cos\theta=1,$$

if $\beta_0<2n/(1+n^2)$ and

$$\cos\theta=-1,$$

if $\beta_0>2n/(1+n^2)$, that is, the photon with a maximal energy flies in the forward direction, while a recoil charge moves either in the forward or reverse direction depending of whether $\beta_0<2n/(1+n^2)$ or $\beta_0>2n/(1+n^2)$.

In the subsequent consideration, we frequently use $\tilde\epsilon=\epsilon/\gamma_0$ instead of ϵ. Then,

$$\cos\theta_\gamma=\frac{1}{\beta_0 n}\left[1+\frac{\tilde\epsilon(n^2-1)}{2}\right],\quad 0<\tilde\epsilon<\tilde\epsilon_{\rm right}=\frac{2(\beta_0 n-1)}{n^2-1}. \tag{2.9}$$

Equations (2.6)–(2.9) can be realized only for $n>1$, $\beta_0>1/n$.

It follows from (2.9) that, in the absence of medium dispersion, photons with different energies $\tilde\epsilon$ fly at different angles θ_γ. Low energy ($\tilde\epsilon\approx 0$) and high energy ($\tilde\epsilon\approx\tilde\epsilon_{\rm right}$) photons fly at the Cherenkov angle $\theta_{\rm c}$ and in the forward direction ($\theta_\gamma=0$), respectively.

A particularly interesting case corresponds to $n\approx 1$ (gases). Putting $n=1+\delta n$, $\beta_0=1-\delta\beta$, we rewrite (2.9) in the form

$$\cos\theta_\gamma=1-\delta n(1-\tilde\epsilon)+\delta\beta,\quad 0<\tilde\epsilon<\tilde\epsilon_{\rm right}\equiv 1-\delta\beta/\delta n. \tag{2.10}$$

Note that always $\delta\beta<\delta n$ (due to the relation $\beta n>1$). It follows from this that $\cos\theta_\gamma$ is enclosed in a narrow interval ($1-\delta n+\delta\beta<\cos\theta_\gamma<1$) to which there corresponds a frequency interval defined in (2.10). For $\delta\beta=\delta n$, one obtains $\tilde\epsilon=0$,

$\cos\theta_\gamma = 1$, that is, zero energy photons fly in the forward direction. For $\delta\beta = 0$, one finds $0 < \tilde{\epsilon} < 1$, $1 - \delta n < \cos\theta_\gamma < 1$, i.e. a very narrow angular interval is filled by the radiation with energy extending from the deep infrared up to the hard Roentgen.

In the past, the expression (2.7) for $\cos\theta_\gamma$ was found by Ginzburg (1940) who recognized that deviation of $\cos\theta_\gamma$ from the Cherenkov value $\cos\theta_c$ is small for the visible and ultraviolet parts of the radiation spectrum.

3. The electronic one-photon Cherenkov effect

(a) The probability of the photon emission

In the framework of quantum electrodynamics, the probability of the one-photon process (electron 1 emits electron 2 and γ quantum) with account taken of all polarizations, is given by (e.g. chapter 3 of Akhiezer & Berestetzky 1969)

$$|N|^2 = -(\bar{u}_2\gamma_\mu u_1)(\bar{u}_1\gamma_\nu u_2)e_\mu^* e_\nu.$$

In explicit form this expression looks rather bulky (Dogyust & Stepanovsky 1968):

$$\begin{aligned}|N|^2 =&- (\bar{u}_2\gamma_\mu u_1)(\bar{u}_1\gamma_\nu u_2)e_\mu^* e_\nu\\ =&- (m^2 + p_1p_2) + (p_2e^*)(p_1e) + (p_1e^*)(p_2e) + m(\{e^*ep_2s_2\} - \{e^*ep_1s_2\}\\ &- \{e^*ep_1s_1\} + \{e^*ep_2s_1\}) - [(m^2 + p_2p_1)(s_2s_1) - (p_2s_1)(p_1s_2)]\\ &+ (m^2 + p_2p_1)[(s_2e^*)(s_1e) + (s_1e^*)(s_2e)] + (s_2s_1)[(p_2e^*)(p_1e) + (p_1e^*)(p_2e)]\\ &- (p_1s_2)[(p_2e^*)(s_1e) + (s_1e^*)(p_2e)] - (p_2s_1)[(p_1e^*)(s_2e) + (s_2e^*)(p_1e)].\end{aligned} \tag{3.1}$$

Here, e is the unit four-vector of the photon polarizations, p_1 and p_2 are electron four-momenta, s_1 and s_2 are the four-vectors of the electron polarization (chapter 2 of Akhiezer & Berestetzky 1969); $abcd$ means $\epsilon_{\mu\nu\sigma\rho}a^\mu b^\nu c^\sigma d^\rho$; the signature is $(+++-)$.

The probability of the one-photon radiation with inclusion of all polarizations is

$$\mathrm{d}w^{(1)} = \frac{e^2}{8\pi}\frac{|N|^2}{E_1p_1}\mathrm{d}t\,\mathrm{d}\omega\,\mathrm{d}\phi. \tag{3.2}$$

If the initial electron is non-polarized and the polarization of the final electron is not fixed, then we should sum over final electron polarizations. For this, we should put $s_1 = s_2 = 0$ in (3.1) and double the remaining terms:

$$\overline{|N|^2} = -2(m^2 + (p_1p_2)) + 2[(p_1e^*)(p_2e) + (p_2e^*)(p_1e)]. \tag{3.3}$$

If the vector of the photon polarization lies in the radiation plane, then

$$\overline{|N|^2} = -2(m^2 + (p_1p_2)) + 4(\boldsymbol{p}_1\boldsymbol{e})(\boldsymbol{p}_2\boldsymbol{e}) = -2(m^2 + (p_1p_2)) + 4p_1^2\sin^2\theta.$$

Here, $\theta \equiv \theta_\gamma$, $v_1 \equiv v_0$, $m \equiv m_0$. For the photon polarization vector perpendicular to the radiation plane

$$\overline{|N|^2} = -2(m^2 + (p_1p_2)).$$

Summing over the photon polarizations, one gets

$$\langle\overline{|N|^2}\rangle = -4(m^2 + (p_1 p_2)) + 4p_1^2 \sin^2\theta = 4p_1^2 \sin^2\theta + 2\omega^2(n^2 - 1). \tag{3.4}$$

Finally, we get the following expression for the photon radiation probability (which includes relativistic quantum corrections) averaged over the polarizations of the initial electron and summed over the polarizations of the emitted photon and recoil electron

$$\begin{aligned} \mathrm{d}w^{(1)} &= e^2\left[v_1 \sin^2\theta + \frac{(n^2-1)\omega^2}{2E_1 p_1}\right]\mathrm{d}t\,\mathrm{d}\omega \\ &= e^2\left\{v_1\left[1 - \frac{1}{v_1^2 n^2}\left(1 + \frac{\omega}{2E_1}(n^2-1)\right)^2\right] + \frac{n^2-1}{2}\,\frac{\omega^2}{E_1 p_1}\right\}\mathrm{d}t\,\mathrm{d}\omega. \end{aligned} \tag{3.5}$$

The probability of the photon radiation per unit of length and per frequency unit is

$$\begin{aligned} \frac{\mathrm{d}^2 w^{(1)}}{\mathrm{d}l\,\mathrm{d}\omega} &= \frac{e^2}{\hbar c^2}\left[1 - \frac{1}{n^2\beta_0^2}\left(1 + \tilde{\epsilon}\frac{n^2-1}{2}\right)^2 + \tilde{\epsilon}^2\,\frac{n^2-1}{2\beta_0^2}\right] \\ &= \frac{e^2}{\hbar c^2}\left(1 - \frac{1}{n^2\beta_0^2} - \frac{n^2-1}{n^2\beta_0^2}\tilde{\epsilon} + \frac{n^4-1}{4n^2\beta_0^2}\tilde{\epsilon}^2\right), \quad \beta_0 = \frac{v_0}{c}. \end{aligned} \tag{3.6}$$

Equations (3.1)–(3.5) were written out in $\hbar = c = 1$ units. The usual dimension is restored in equation (3.6) and the subsequent ones.

(*b*) *The radiated energy*

The energy radiated per unit length and per frequency unit is obtained by multiplying (3.6) by $\hbar\omega$:

$$\frac{\mathrm{d}^2 E}{\mathrm{d}l\,\mathrm{d}\omega} = \frac{e^2 m_0 \gamma_0}{\hbar}\tilde{\epsilon}\left[1 - \frac{1}{n^2\beta_0^2}\left(1 + \tilde{\epsilon}\frac{n^2-1}{2}\right)^2 + \tilde{\epsilon}^2\,\frac{n^2-1}{2\beta_0^2}\right]. \tag{3.7}$$

Equations (2.7), (2.9), (3.7) and (4.1) are valid for arbitrary dependence $n(\omega)$. We are interested in studying the role of quantum correction terms. To separate the contribution of dispersive effects, we disregard the ω dependence of the refractive index in the numerical estimations following (3.7) and (4.1). The total radiated energy is finite. It is obtained by integrating (3.7) over ω

$$\frac{\mathrm{d}E}{\mathrm{d}l} = \frac{e^2 m_0^2 \gamma_0^2 c^2}{\hbar^2}\left[\left(1 - \frac{1}{n^2\beta_0^2}\right)\frac{\tilde{\epsilon}^2_{\mathrm{right}}}{2} - \frac{n^2-1}{3n^2\beta_0^2}\tilde{\epsilon}^3_{\mathrm{right}} + \frac{n^4-1}{16n^2\beta_0^2}\tilde{\epsilon}^4_{\mathrm{right}}\right].$$

The Tamm–Frank radiation intensity

$$\left(\frac{\mathrm{d}^2 E}{\mathrm{d}l\,\mathrm{d}\omega}\right)_{\mathrm{TF}} = \frac{e^2 m_0 \gamma_0}{\hbar}\tilde{\epsilon}\left(1 - \frac{1}{n^2\beta_0^2}\right)$$

is obtained from (3.7) by dropping terms with $\tilde{\epsilon}$ inside the square brackets. As a function of $\tilde{\epsilon}$, (3.7) (in the absence of medium dispersion) equals zero at $\epsilon = 0$ and smoothly rises everywhere if

$$\frac{1}{n}\sqrt{1+\frac{4}{3}\,\frac{n^2-1}{n^2+1}}<\beta_0<1. \tag{3.8}$$

For

$$\beta_0<\frac{1}{n}\sqrt{1+\frac{4}{3}\,\frac{n^2-1}{n^2+1}}, \tag{3.9}$$

(3.7) has a maximum at

$$\tilde{\epsilon}_{\max}=\frac{4}{3(n^2+1)}(1-R)$$

and a minimum at

$$\tilde{\epsilon}_{\min}=\frac{4}{3(n^2+1)}(1+R).$$

Here

$$R=\sqrt{1-\frac{3}{4}\,\frac{n^2+1}{n^2-1}(\beta_0^2n^2-1)}.$$

It is easy to check that $\tilde{\epsilon}_{\min}$ always lies in a kinematically forbidden region while $\tilde{\epsilon}_{\max}$ lies in a kinematically permissible region for

$$\frac{1}{n}<\beta_0<\frac{3n}{2n^2+1} \tag{3.10}$$

and in a kinematically forbidden region for

$$\frac{3n}{2n^2+1}<\beta_0<\frac{1}{n}\sqrt{1+\frac{4}{3}\,\frac{n^2-1}{n^2+1}}. \tag{3.11}$$

It follows from (3.8)–(3.11) that for

$$\frac{1}{n}<\beta_0<\frac{3n}{2n^2+1} \tag{3.12}$$

the radiation intensity (3.7) has a maximum in a kinematically permissible region while for

$$\frac{3n}{2n^2+1}<\beta_0<1 \tag{3.13}$$

it continuously rises there reaching the maximum value at the right border of the kinematically permissible region lying on the axis of the Cherenkov cone.

We estimate now the ratio of (3.7) taken at the right border to the Tamm–Frank radiation intensity taken at the typical optical frequency $\omega_{\rm opt}=6\times10^{15}\,{\rm s}^{-1}$. Let the velocity be very close to 1: $\beta_0=1-\delta\beta$ and $n=2$ (this corresponds to (3.13)). Then, the ratio just mentioned is

$$\frac{16}{27}\,\frac{m_0c^2}{\hbar\omega_{\rm opt}\sqrt{2\delta\beta}}.$$

For the chosen optical frequency, the photon energy $\hbar\omega_{\rm opt} \approx 4\,{\rm eV}$. Taking for $m_0c^2 = 0.51$ MeV (electron) one gets for this ratio $4\times 10^4/\sqrt{\delta\beta}$. The corresponding photon energy $\hbar\omega_{\rm right} \approx 0.24/\sqrt{\delta\beta}$ MeV is much larger than the typical energy of optical electrons (few electron-volts). This means that the radiation intensity is very large on the axis of the Cherenkov cone. In the past, the possibility of the one-photon radiation in the forward direction by a charge moving fast in medium was suggested by Tyapkin (1993) on purely intuitive grounds.

To make a rough estimate for the angle and energy corresponding to the maximum of the radiation intensity (at $\tilde{\epsilon}_{\max}$), consider concrete values $n=2$ and $\beta_0=7/12$ (this corresponds to (3.12)). Then, $\theta_{\max}\approx 22°$ and $\hbar\omega_{\max} \approx 4\times 10^4$ eV. Similar values for the frequency $\omega_{\rm opt}=6\times 10^{15}\,{\rm s}^{-1}$ lying within the visible light region are $\theta_{\rm opt}\approx 31°$ and $\hbar\omega_{\rm opt}\approx 4\,{\rm eV}$. This means that a strong Roentgen radiation will be observed at $\theta_{\max}\approx 22°$. The ratio of (3.7) at the maximum (i.e. at 22°) to (3.7) taken for the optical frequency $\omega_{\rm opt}$ (i.e. at 31°) is $\approx 5\times 10^3$. This means that the radiation intensity at the maximum is 5×10^3 larger than in an optical region. The value of $\mathrm{d}^2E/\mathrm{d}l\,\mathrm{d}\omega$ at the right border of the kinematically permissible region,

$$\frac{\mathrm{d}^2E}{\mathrm{d}l\,\mathrm{d}\omega} = \frac{e^2 m_0\gamma_0}{\hbar}\tilde{\epsilon}^{\,3}_{\rm right}\frac{n^2-1}{2\beta_0^2}, \tag{3.14}$$

is smaller than its value at the maximum ($\tilde{\epsilon}_{\rm right}$ is the same as in (2.9)).

(i) *Application to gases*

We evaluate the radiation intensity for gases ($n=1+\delta n$, $\beta_0=1-\delta\beta$, $\delta\beta<\delta n$, $\delta\beta\ll 1$, $\delta n\ll 1$):

$$\frac{\mathrm{d}^2E}{\mathrm{d}l\,\mathrm{d}\omega} = \frac{e^2 m_0\delta n}{\hbar\sqrt{2\delta\beta}}\tilde{\epsilon}[2(1-\delta\beta/\delta n)+\tilde{\epsilon}(\tilde{\epsilon}-2)]. \tag{3.15}$$

It is seen that the radiation intensity, despite its smallness (due to the factor δn) is much larger in the Roentgen part of the radiation spectrum than in the optical one. We now estimate (3.15) at the maximum. After some algebra one gets

$$\frac{\mathrm{d}^2E}{\mathrm{d}l\,\mathrm{d}\omega} = \frac{e^2 m_0\delta n}{2\hbar\sqrt{2\delta\beta}}\left(1-\frac{\delta\beta}{\delta n}\right)^2\left[1+\frac{1}{4}\left(1-\frac{\delta\beta}{\delta n}\right)\right]. \tag{3.16}$$

(ii) *Comparison with Ginzburg's results*

Ginzburg (1940) considered only the non-relativistic limit ($\beta_0\ll 1$, $n\gg 1$). Making this approximation in (3.7), we find that the Ginzburg formula (29) still differs from ours by the last term in (3.7) which corresponds to the electron magnetic moment contribution. The reason is that Ginzburg supposed this contribution to be negligible (this is true for small recoil energies considered in Ginzburg 1940). In fact, adding formulae (36) and (37) from Ginzburg (1940) and setting in them $\mu_0=e\hbar/2mc^2$, we get the non-relativistic version of (3.7). Unfortunately, no analysis was made in Ginzburg (1940) of the energy and angular distributions of radiation for large energy–momentum transfer inside the kinematically permissible region. This is the main goal of the present consideration.

4. One-photon Cherenkov effect for spinless charges

The situation is simplified for the spinless charged particles. The energy radiated per unit length and unit frequency is given by

$$\frac{\mathrm{d}^2 E}{\mathrm{d}l\,\mathrm{d}\omega} = \frac{e^2 m_0 \gamma_0}{\hbar}\,\tilde{\epsilon}\left[1 - \frac{1}{n^2\beta_0^2}\left(1 + \tilde{\epsilon}\,\frac{n^2-1}{2}\right)^2\right]. \tag{4.1}$$

The non-relativistic version of (4.1) coincides with eqn (29) of Ginzburg (1940) (since the magnetic moment contribution was disregarded there).

It is seen that (4.1) vanishes for $\tilde{\epsilon}=0$ and for $\tilde{\epsilon}=\tilde{\epsilon}_{\max}$ and takes the maximal value at

$$\tilde{\epsilon} = \frac{4}{3(n^2-1)}(R-1),$$

where $R=\sqrt{1+3(n^2\beta_0^2-1)/4}$. For rough estimates we again take the same $\beta_0=7/12$, $m_0c^2=0.51$ Mev and $n=2$. Then, the angle corresponding to the maximum of (3.11) (at $\tilde{\epsilon}\approx 0.057\hbar\omega_{\max}\approx 3.6\times 10^4$ eV) is $\theta_{\max}=21°$, and the angle corresponding to the optical frequency $\omega_{\mathrm{opt}}=6\times 10^{15}\ \mathrm{s}^{-1}$ is $\theta_{\mathrm{opt}}=31°$. The ratio of radiation intensities for these angles is $\approx 5.1\times 10^3$. Again we see that the radiation intensity in the maximum lying in the Roentgen part of the radiation spectrum is much larger than the radiation intensity in its optical part where the observations are usually made.

5. Discussion

It follows from the previous consideration that the VC radiation fills the interior of the Cherenkov cone ($\cos\theta_\gamma=1/\beta_0 n$). The angular region of the VC radiation corresponding to the visible light is surrounded by the angular region of the infrared VC radiation adjoining the Cherenkov cone and by the angular region of the high energy VC radiation ranging up to small angles.

Up to now, we disregarded the ω dependence of the refractive index n. In this case, to each value of ω there corresponds the only angle θ. That is, photons with different energies fly at different angles θ. The situation essentially changes when the medium dispersion is taken into account. In general, the dependence $n(\omega)$ contains the broad absorption bands with rather narrow transparent regions between them. There will be no photons with ω lying within the absorption bands and no photons at the angles corresponding to these ω. On the other hand, relation (2.9) is still valid for photons with ω lying within the transparent regions where $n>1$. As an example, consider the highly idealized case when the transparent regions present the sum of δ functions: $n(\omega)=\sum n_i\delta(\omega,\omega_i)$. Then, the angular spectrum will consist of infinitely thin concentric rings at angles defined by (2.9) where instead of $\tilde{\epsilon}$ and n one should substitute $\tilde{\epsilon}_i$ and n_i. The real transparency bands have a finite width and, therefore, finite width angular rings (which can overlap between themselves) should be observed.

(a) Comparison with experiment

In our opinion, the main reason why the infrared and Roentgen parts of the radiation spectrum were not observed is that experimentalists use the optical and detecting devices operating in the visible part of the radiation spectrum $(4\text{–}6)\times 10^{-5}$ cm.

Yet, there are implicit experimental indications. We briefly enumerate them.

(i) In Vodopjanov *et al.* (2000), the narrow Cherenkov ring originating from the VC radiation of Pb^{208} ions was observed on the white and black photographic film. This ring was surrounded by two broad rings of the lower intensity. Vodopjanov *et al.* (2000) were unable to explain their origin. We associate them with infrared and high energy parts of the VC radiation filling the interior of the Cherenkov cone. However, some caution is needed. When evaluating the photon emission by the charge moving in medium with a superluminal velocity we put the charge formfactor to be equal to 1, thus considering the charge as structureless. This is not true for heavy ions, especially for large energy–momentum transfer. This means that formulae obtained in §§3 and 4 describe the VC radiation of heavy ions only qualitatively. These questions are discussed in chapter 7 of Afanasiev (2004).

(ii) In Stevens *et al.* (2001) and Wahlstrand & Merlin (2003), the striped structure of the radiation intensity was observed inside the Cherenkov cone (see their fig. 2). Wahlstrand & Merlin (2003) associated it with the medium dispersion. We think that this is only partly true. It follows from the present consideration that both dispersion and quantum correction terms contribute to the above striped structure.

6. Conclusion

We investigated the fine structure of the radiation field arising from the charge uniform unbounded motion with the velocity greater than the light velocity in medium (that is the VC radiation field). It turns out that for the constant refractive index the radiation field fills the entire Cherenkov cone, not only its surface (as it is usually believed). In the absence of dispersion, photons with different energies fly at different angles: the photons with small and very high energies fly at the Cherenkov angle and in the direction of the initial charge, respectively. Inside the Cherenkov cone, the radiation intensity reaches the maximal value in the Roentgen part of the radiation spectrum, in accordance with Vavilov's predictions. The inclusion of the medium dispersion leads to the appearance of the striped-like radiation structure (the family of concentric Cherenkov rings) inside the Cherenkov cone.

Where can these results be applied? VC radiation is frequently used for identification of the charge velocity (e.g. in neutrino experiments). If the Cherenkov photon flies at the angle θ, then, using the Tamm–Frank formula (1.1), one extracts the charge velocity: $\beta_0 = 1/n\cos\theta$. However, this formula gives wrong results for the high-energy photon. In this case equation (2.9) containing quantum correction terms should be resolved w.r.t. β_0.

After this paper was finished, we became aware of two papers having some overlap with this one. Equations (3.3) and (3.7) were obtained in Jauch & Watson (1948) and Sokolow (1940), respectively. Correspondingly, the present paper concerns only the physical consequences following from these equations.

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