The isomorphism problem for finitary incidence rings

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The notion of a finitary incidence algebra was first introduced in [1] as a generalization of the notion of an incidence algebra for the case of an arbitrary poset. It was shown that the isomorphism problem for such algebras was solved positively ([1], Theorem 5). In the present talk we consider this problem in more general case.

Let $\mathcal{C}$ be a preadditive small category. Assume that the binary relation $\leq$ on the set of its objects, such that $x \leq y \iff Mor(x,y) \neq 0$, is a partial order. Consider the set of formal sums of the form

$$\alpha = \sum_{x \leq y} \alpha_{xy}[x,y], \tag{1}$$

where $\alpha_{xy} \in Mor(x,y)$, $[x,y]$ is a segment of the partial order. A formal sum (1) is called *a finitary series*, if for any $x, y \in Ob\,\mathcal{C}$, $x < y$ there exists only a finite number of $[u,v] \subset [x,y]$, such that $u < v$ and $\alpha_{uv} \neq 0$. The set of the finitary series is denoted by $FI(\mathcal{C})$.

The addition of the finitary series is inherited from the addition of the morphisms. The multiplication is defined by means of the convolution:

$$\alpha\beta = \sum_{x \leq y} \left(\sum_{x \leq z \leq y} \alpha_{xz}\alpha_{zy} \right) [x,y],$$

where $\alpha_{xz}\alpha_{zy} = \alpha_{zy} \circ \alpha_{xz} \in Mor(x,y)$. Under these operations $FI(\mathcal{C})$ form an associative ring with identity, which is called *a finitary incidence ring of a category.*

It is easy to see, that the description of the idempotents of $FI(\mathcal{C})$ can be obtained as in [1]. This allows us to solve the isomorphism problem for finitary incidence rings of preorders.

Let R be an associative ring with identity, $P(\preccurlyeq)$ an arbitrary preordered set. Denote by $\sim$ the equivalence relation on P, such that $x \sim y$ iff $x \preccurlyeq y$ and $y \preccurlyeq x$. Define $M([x],[y])$ to be an abelian group of row and column finite matrices over R, indexed by the elements of the equivalence classes $[x]$ and $[y]$. Consider the following preadditive category $\mathcal{C}$:

1. $Ob\,\mathcal{C} = P/_{\sim}$ with the induced order $\leq$;
2. For any pair $[x],[y] \in Ob\,\mathcal{C}$ define $Mor([x],[y]) = M([x],[y])$, if $[x] \leq [y]$, and 0 otherwise (the composition of the morphisms is the matrix multiplication).

Denote the finitary incidence ring of this category by $FI(P,R)$. Obviously, $FI(P,R)$ is an algebra over R, which is called *a finitary incidence algebra of P over R.*

Theorem 1. *Let P and Q be preordered sets, R and S indecomposable commutative rings with identity, $\mathcal{C}$ and $\mathcal{D}$ preadditive categories corresponding to the pairs (P, R) and (Q, S), respectively. If $FI(P, R) \cong FI(Q, S)$ as rings, then $\mathcal{C} \cong \mathcal{D}$.*

Corollary 1. *Let P and Q be class finite preordered sets, R and S indecomposable commutative rings with identity. If $FI(P, R) \cong FI(Q, S)$ as rings, then $P \cong Q$ and $R \cong S$.*

As a corollary we obtain the positive solution of the isomorphism problem for weak incidence algebras given in [2] .

References

[1] N. S. Khripchenko and B. V. Novikov, Finitary incidence algebras, Communications in Algebra, **37**(2009), no. 5, 1670–1676.

[2] S. Singh and F. Al-Thukair, Weak incidence algebra and maximal ring of quotients, Int.J.Math. Math. Sci. 2004 (2004), no. 53, 2835–2845.

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