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About One Model of Consecutive Qubit Binary Testing

M. Thawi, G. M. Zholtkevych

V. N. Karazin Kharkiv National University, Ukraine

Qubit binary tests models are considered in the paper. For each binary test the state of compound quantum-classical system is associated. Formulas for density matrices transformation under binary test are obtained. Method for computing probabilistic characteristics of testing outcomes is proposed. Class of probabilistic measures on testing outcome sequences space is described.

Key words: *qubit, quantum measurement, projective measurement, consecutive quantum measurement.*

В статье рассмотрены модели двоичных тестов для кубита. Показано, что с каждым тестом связано определенное состояние составной квантово-классической системы. Получен вид преобразований матриц плотности состояний кубита в результате тестирования. Предложен метод вычисления вероятностных характеристик результатов тестирования. Описан класс вероятностных мер на пространстве последовательностей двоичных исходов, которые возникают в результате тестирования.

Ключевые слова: *кубит, квантовое измерение, проекционное измерение, последовательное квантовое измерение.*

В статі розглянуті моделі бінарних тестів для кубіту. Показане, що з кожним тестом пов'язаний певний стан складеної квантово-класичної системи. Наведений вид перетворення матриць щільності станів кубіту в наслідок тестування. Запропонований метод обчислення ймовірнісних характеристик результатів тестування. Описано клас імовірнісних мір на просторі послідовностей бінарних наслідків, що виникають в процесі тестування.

Ключові слова: *кубіт, квантове вимірювання, проекційне вимірювання, послідовне квантове вимірювання.*

Introduction

Nowadays, quantum computing is being considered as a prospective research area for solving computing complexity problem [1, 2]. There are great expectations that a capability of quantum computing systems is streets ahead of a capability of classical computing systems. Research results [3 – 5] corroborate these expectations.

The cardinal problem in this research area is modeling problem for quantum information processes. As known, an assertion and an operator are key concepts of a mathematical model of classical computing processes [6]. In the case of quantum computing process, assertions are changed by measuring procedures and operators are changed by dynamical transformations [7]. In this paper we consider models of quantum measuring procedures for some class – measurement with two outcomes. We call such measuring procedures by quantum binary tests. The paper objective is to study such measuring procedure for 2-level quantum systems (qubits).

Note that we do not restrict our studying by standard (projective) quantum measuring procedures. It ensures a possibility of taking into account a new class of information signal detectors based on Josephson solid-state qubits [8]. Research frame extension requires considering consecutive quantum measuring procedures.

Our objectives are

1) to develop models of a qubit binary test and a process of a consecutive measurement of a qubit using such tests;
2) to study developing models.

1. Models of qubit states

Let $\mathcal{H}_2$ be a 2-dimensional Hilbert space of a qubit.

It is well-known [9, 10] that a non-negative operator ρ in $\mathcal{H}_2$ is a model of a qubit state if it holds condition $\mathrm{Tr}(\rho)=1$. In this case ρ is called a density matrix.

Density matrix ρ corresponds to a pure state if it is a one-dimensional ortho-projector, i.e. $\rho = |\xi\rangle\langle\xi|$ for some unit vector ξ in $\mathcal{H}_2$. In other case density matrix ρ corresponds to a mixed state.

2. Models of qubit binary tests

We shall use the following definition for qubit binary test [10]: let $\mathbf{M} = \{M_0, M_1\}$ be a pair of non-negative operators in $\mathcal{H}_2$, it is called a qubit binary test if the operators satisfy condition $M_0 + M_1 = \mathbf{1}$.

Hence, we can characterize qubit binary test $\mathbf{T}$ by an operator M in $\mathcal{H}_2$, where $\mathbf{0} < M < \mathbf{1}$. Operators M_0 and M_1 can be restored by formulas $M_0 = \mathbf{1} - M$ and $M_1 = M$.

Test $\mathbf{T}$ defines an affine map from a set of all states to a set of all probability distributions on a set $\Sigma = \{0,1\}$:

$$\begin{cases} \mathrm{Pr}_\rho^{\mathbf{T}}(0) = \mathrm{Tr}(\rho \cdot (\mathbf{1} - M)) \\ \mathrm{Pr}_\rho^{\mathbf{T}}(1) = \mathrm{Tr}(\rho \cdot M) \end{cases} \tag{2.1}$$

3. Naimark's theorem for qubit binary tests

In this and next sections we shall obtain an explicit relation between qubit binary tests and projective (von Neumann's) measurements with two outcomes [9]. Existence of this relation follows from Naimark's theorem [11], but we need explicit representations of all elements of Naimark's construction.

<u>Statement 3.1.</u> If $\mathbf{T}$ is a qubit binary test, M is an operator in $\mathcal{H}_2$ such that condition $\mathbf{0} < M < \mathbf{1}$ is held then there exist ortho-projectors E_0 and E_1 in $\mathcal{H}_2 \otimes \mathcal{H}_2$ and an isometry $V : \mathcal{H}_2 \to \mathcal{H}_2 \otimes \mathcal{H}_2$ such that the following conditions hold:

$$E_0 + E_1 = \mathbf{1},\ E_0 \cdot E_1 = \mathbf{0} \tag{3.1}$$

$$\mathbf{1} - M = V^* E_0 V,\ M = V^* E_1 V \tag{3.2}$$

Proof. Let $\{|0\rangle, |1\rangle\}$ be an ortho-normal basis in $\mathcal{H}_2$ and $|0\rangle, |1\rangle$ are eigenvectors of the operator M corresponding to eigenvalues $m_0 \le m_1$ respectively. In this case the operator M is represented by the following formula:

$$M = m_0 |0\rangle\langle 0| + m_1 |1\rangle\langle 1| \tag{3.3}$$

Consider an operator $V:\mathcal{H}_2 \to \mathcal{H}_2 \otimes \mathcal{H}_2$ such that

$$V|0\rangle = \sqrt{1-m_0}\,|00\rangle + \sqrt{m_0}\,|01\rangle$$
$$V|1\rangle = \sqrt{1-m_1}\,|10\rangle + \sqrt{m_1}\,|11\rangle$$

Let $a_0(x)$ be equal to $\sqrt{1-x}$ and $a_1(x)$ be equal to $\sqrt{x}$.

Then we can rewrite for $s \in \{0,1\}$

$$V|s\rangle = \sum_{s'\in\{0,1\}} a_{s'}(m_s)|ss'\rangle \tag{3.4}$$

Note, that for $\xi \in \mathcal{H}_2$

$$V\xi = V(|0\rangle\langle 0|\xi\rangle + |1\rangle\langle 1|\xi\rangle) = \langle 0|\xi\rangle\sqrt{1-m_0}\,|00\rangle + \langle 0|\xi\rangle\sqrt{m_0}\,|01\rangle +$$
$$+\langle 1|\xi\rangle\sqrt{1-m_1}\,|10\rangle + \langle 1|\xi\rangle\sqrt{m_1}\,|11\rangle$$

and

$$\|V\xi\|^2 = (1-m_0)|\langle 0|\xi\rangle|^2 + m_0|\langle 0|\xi\rangle|^2 + (1-m_1)|\langle 1|\xi\rangle|^2 + m_1|\langle 1|\xi\rangle|^2$$
$$= |\langle 0|\xi\rangle|^2 + |\langle 1|\xi\rangle|^2 = \|\xi\|^2$$

Therefore V is an isometry.

We obviously have for each $s \in \{0,1\}$

$$\langle s|V^*|s's''\rangle = \overline{\langle s's''|V|s\rangle} = \sum_{s^{(3)}\in\{0,1\}} a_{s^{(3)}}(m_s)\overline{\left\langle s's''\middle|ss^{(3)}\right\rangle} =$$
$$= \sum_{s^{(3)}\in\{0,1\}} a_{s^{(3)}}(m_s)\left\langle ss^{(3)}\middle|s's''\right\rangle = a_{s''}(m_s)\langle s|s'\rangle = a_{s''}(m_{s'})\langle s|s'\rangle$$

Therefore,

$$V^*|s's''\rangle = a_{s''}(m_{s'})|s'\rangle \tag{3.5}$$

Using (3.5), we get

$$\begin{aligned} V^*|00\rangle &= \sqrt{1-m_0}\,|0\rangle \qquad & V^*|10\rangle &= \sqrt{1-m_1}\,|1\rangle \\ V^*|01\rangle &= \sqrt{m_0}\,|0\rangle \qquad & V^*|11\rangle &= \sqrt{m_1}\,|1\rangle \end{aligned} \tag{3.6}$$

Denote by E_0 and E_1 the following ortho-projectors

$$\begin{aligned} E_0 &= |00\rangle\langle 00| + |10\rangle\langle 10| \\ E_1 &= |01\rangle\langle 01| + |11\rangle\langle 11| \end{aligned} \tag{3.7}$$

Using (3.5) and (3.6), we get

$$V^*E_0V|s\rangle = V^*(|00\rangle\langle 00| + |10\rangle\langle 10|)V|s\rangle =$$
$$= V^*(|00\rangle\langle 00| + |10\rangle\langle 10|)\sum_{s'\in\{0,1\}} a_{s'}(m_s)|ss'\rangle =$$
$$= V^*\sum_{s'\in\{0,1\}} a_{s'}(m_s)(|00\rangle\langle 00|ss'\rangle + |10\rangle\langle 10|ss'\rangle) =$$
$$= V^*\left(a_0(m_s)|00\rangle\langle 0|s\rangle + a_0(m_s)|10\rangle\langle 1|s\rangle\right) =$$

$$= a_0(m_s)\left(\sqrt{1-m_0}\,|0\rangle\langle 0|s\rangle + \sqrt{1-m_1}\,|1\rangle\langle 1|s\rangle\right) =$$
$$= a_0(m_s)a_0(m_0)|0\rangle\langle 0|s\rangle + a_0(m_s)a_0(m_1)|1\rangle\langle 1|s\rangle$$

Hence,

$$V^*E_0V = \sum_{s\in\{0,1\}} V^*E_0V|s\rangle\langle s| =$$
$$= \sum_{s\in\{0,1\}} \left(a_0(m_s)a_0(m_0)|0\rangle\langle 0|s\rangle\langle s| + a_0(m_s)a_0(m_1)|1\rangle\langle 1|s\rangle\langle s|\right) =$$
$$= a_0(m_0)a_0(m_0)|0\rangle\langle 0| + a_0(m_1)a_0(m_1)|1\rangle\langle 1| =$$
$$= (1-m_0)|0\rangle\langle 0| + (1-m_1)|1\rangle\langle 1| = \mathbf{1} - \left(m_0|0\rangle\langle 0| + m_1|1\rangle\langle 1|\right) = \mathbf{1} - M$$

Therefore,

$$V^*E_0V = \mathbf{1} - M$$

Similarly,

$$V^*E_1V = M$$

This completes the proof of Statement 3.1. QED.

4. Qubit binary tests and projective measurements of combined system

In this section we shall establish that a qubit binary test can be considered as a sequence of three steps: the first, combining qubit with a classical system (an instrument); the second, measuring the combined system by some projective measurement with two outcomes; and the third, removing the instrument from the combined system.

Statement 4.1. Let $\mathbf{T} = \{\mathbf{1} - M, M\}$ be a qubit binary test, ρ be a qubit state density matrix, V be an isometry related with $\mathbf{T}$ according to statement 2.1 then $V\rho V^*$ is a density matrix.

Proof. Evidently, that $V\rho V^*$ is a non-negative operator in $\mathcal{H}_2 \otimes \mathcal{H}_2$.

By direct calculation we can obtain:

$$V\rho V^* = V\left(|0\rangle\langle 0|\rho|0\rangle\langle 0| + |0\rangle\langle 0|\rho|1\rangle\langle 1| + |1\rangle\langle 1|\rho|0\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1|\right)V^* =$$
$$= V|0\rangle\langle 0|\rho|0\rangle\langle 0|V^* + V|0\rangle\langle 0|\rho|1\rangle\langle 1|V^* + V|1\rangle\langle 1|\rho|0\rangle\langle 0|V^* + V|1\rangle\langle 1|\rho|1\rangle\langle 1|V^* =$$
$$= \left(\sqrt{1-m_0}\,|00\rangle + \sqrt{m_0}\,|01\rangle\right)\langle 0|\rho|0\rangle\left(\sqrt{1-m_0}\,\langle 00| + \sqrt{m_0}\,\langle 01|\right) +$$
$$+ \left(\sqrt{1-m_0}\,|00\rangle + \sqrt{m_0}\,|01\rangle\right)\langle 0|\rho|1\rangle\left(\sqrt{1-m_1}\,\langle 10| + \sqrt{m_1}\,\langle 11|\right) +$$
$$+ \left(\sqrt{1-m_1}\,|10\rangle + \sqrt{m_1}\,|11\rangle\right)\langle 1|\rho|0\rangle\left(\sqrt{1-m_0}\,\langle 00| + \sqrt{m_0}\,\langle 01|\right) +$$
$$+ \left(\sqrt{1-m_1}\,|10\rangle + \sqrt{m_1}\,|11\rangle\right)\langle 1|\rho|1\rangle\left(\sqrt{1-m_1}\,\langle 10| + \sqrt{m_1}\,\langle 11|\right)$$

Transposing members of this equality we get

$$V\rho V^* = \langle 0|\rho|0\rangle\left((1-m_0)|00\rangle\langle 00| + \sqrt{m_0(1-m_0)}\,|00\rangle\langle 01| + \right. \tag{4.1}$$
$$\left. + \sqrt{m_0(1-m_0)}\,|01\rangle\langle 00| + m_0|01\rangle\langle 01|\right) +$$
$$+ \langle 0|\rho|1\rangle\left(\sqrt{(1-m_0)(1-m_1)}\,|00\rangle\langle 10| + \sqrt{m_1(1-m_0)}\,|00\rangle\langle 11| + \right.$$

$$+\sqrt{m_0(1-m_1)}|01\rangle\langle 10|+\sqrt{m_0 m_1}|01\rangle\langle 11|\Big)+$$
$$+\langle 1|\rho|0\rangle\Big(\sqrt{(1-m_0)(1-m_1)}|10\rangle\langle 00|+\sqrt{m_0(1-m_1)}|10\rangle\langle 01|+$$
$$+\sqrt{m_1(1-m_0)}|11\rangle\langle 00|+\sqrt{m_0 m_1}|11\rangle\langle 01|\Big)+$$
$$+\langle 1|\rho|1\rangle\Big((1-m_1)|10\rangle\langle 10|+\sqrt{m_1(1-m_1)}|10\rangle\langle 00|+$$
$$+\sqrt{m_1(1-m_1)}|11\rangle\langle 10|+m_1|11\rangle\langle 11|\Big)$$

Now it is easily seen that

$$\mathrm{Tr}\left(V\rho V^*\right)=1$$

Hence, $V\rho V^*$ is a density matrix. QED.

Remark 4.1. We shall consider $V\rho V^*$ as a density matrix of a combined system state.

Statement 4.2. Let $\mathbf{T}=\{\mathbf{1}-M,M\}$ be a qubit binary test, ρ be a qubit state density matrix, V, E_0, E_1 be an isometry and ortho-projectors related with $\mathbf{T}$ according to statement 2.1 then the probability distribution of outcomes of projective measurement $\{E_0,E_1\}$ for the combined system state $V\rho V^*$ is

$$\begin{aligned}\Pr_{V\rho V^*}^{\{E_0,E_1\}}(0)&=\mathrm{Tr}(\rho(\mathbf{1}-M))\\ \Pr_{V\rho V^*}^{\{E_0,E_1\}}(1)&=\mathrm{Tr}(\rho M)\end{aligned}\tag{4.2}$$

Proof. As known for projective measurement [9]

$$\Pr_{V\rho V^*}^{\{E_0,E_1\}}(0)=\mathrm{Tr}\left(V\rho V^*E_0\right)=\mathrm{Tr}\left(\rho V^*E_0V\right)$$

Using (3.2) we get

$$\Pr_{V\rho V^*}^{\{E_0,E_1\}}(0)=\mathrm{Tr}(\rho(\mathbf{1}-M))$$

This completes the proof of Statement 4.2. QED.

For an operator $A\otimes B$ in $\mathcal{H}_2\otimes\mathcal{H}_2$ by $\mathrm{Tr}_{\mathrm{ins}}(A\otimes B)$ denote an operator in $\mathcal{H}_2$:

$$\mathrm{Tr}_{\mathrm{ins}}[A\otimes B]=\mathrm{Tr}(B)A$$

The map $\mathrm{Tr}_{\mathrm{ins}}$ corresponds to removing the second part of the combined system.

Statement 4.3. Let $\mathbf{T}=\{\mathbf{1}-M,M\}$ be a qubit binary test, ρ be a qubit state density matrix, V, E_0, E_1 be an isometry and ortho-projectors related with $\mathbf{T}$ according to statement 2.1 then

$$\mathrm{Tr}_{\mathrm{ins}}\left[\frac{E_0V\rho V^*E_0}{\mathrm{Tr}\left(V\rho V^*E_0\right)}\right]=\frac{(\mathbf{1}-M)^{1/2}\rho(\mathbf{1}-M)^{1/2}}{\mathrm{Tr}(\rho(\mathbf{1}-M))}\tag{4.3}$$

and

$$\mathrm{Tr}_{\mathrm{ins}}\left[\frac{E_1 V\rho V^* E_1}{\mathrm{Tr}\left(V\rho V^* E_1\right)}\right]=\frac{M^{1/2}\rho M^{1/2}}{\mathrm{Tr}(\rho M)} \tag{4.4}$$

Proof. Applying to these states the removing map $\mathrm{Tr}_{\mathrm{ins}}$ and using (4.1) and (4.2) we get

$$\mathrm{Tr}_{\mathrm{ins}}\left[\frac{E_0 V\rho V^* E_0}{\mathrm{Tr}\left(V\rho V^* E_0\right)}\right]=\frac{1}{\mathrm{Tr}\left(V\rho V^* E_0\right)}\mathrm{Tr}_{\mathrm{ins}}\left[(1-m_0)\langle 0|\rho|0\rangle|00\rangle\langle 00|+\right.$$

$$+\sqrt{(1-m_0)(1-m_1)}\langle 0|\rho|1\rangle|00\rangle\langle 10|+\sqrt{(1-m_0)(1-m_1)}\langle 1|\rho|0\rangle|10\rangle\langle 00|\Big)+$$

$$\left.+(1-m_1)\langle 1|\rho|1\rangle|10\rangle\langle 10|\right]=\frac{(\mathbf{1}-M)^{1/2}\rho(\mathbf{1}-M)^{1/2}}{\mathrm{Tr}(\rho(\mathbf{1}-M))}$$

Similarly,

$$\mathrm{Tr}_{\mathrm{ins}}\left[\frac{E_1 V\rho V^* E_1}{\mathrm{Tr}\left(V\rho V^* E_1\right)}\right]=\frac{M^{1/2}\rho M^{1/2}}{\mathrm{Tr}(\rho M)}$$

This completes the proof of Statement 4.3. QED.

Remark 4.2. By the von Neumann's postulate [10] if we make projective measurement $\{E_0, E_1\}$ of a quantum system and before measurement its state is described by density matrix $V\rho V^*$ then after the measurement the system state equals to $\dfrac{E_0 V\rho V^* E_0}{\mathrm{Tr}\left(V\rho V^* E_0\right)}$ if measuring outcome has been equal to '0' and equals to $\dfrac{E_1 V\rho V^* E_1}{\mathrm{Tr}\left(V\rho V^* E_1\right)}$ if measuring outcome has been equal to '1'. Statement 4.3 grounds that a qubit state after binary testing equals to $\dfrac{(\mathbf{1}-M)^{1/2}\rho(\mathbf{1}-M)^{1/2}}{\mathrm{Tr}(\rho(\mathbf{1}-M))}$ or $\dfrac{M^{1/2}\rho M^{1/2}}{\mathrm{Tr}(\rho M)}$.

Denote posterior states of a qubit after binary testing by

$$\mathbf{T}_0[\rho]=\frac{(\mathbf{1}-M)^{1/2}\rho(\mathbf{1}-M)^{1/2}}{\mathrm{Tr}(\rho(\mathbf{1}-M))} \tag{4.5}$$

$$\mathbf{T}_1[\rho]=\frac{M^{1/2}\rho M^{1/2}}{\mathrm{Tr}(\rho M)} \tag{4.6}$$

Statement 4.4. Let $\mathbf{T}=\{\mathbf{1}-M, M\}$ be a qubit binary test, ρ be a qubit state density matrix, V, E_0, E_1 be an isometry and ortho-projectors related with $\mathbf{T}$ according to statement 3.1 then

$$\mathrm{Tr}_{\mathrm{ins}}\left[E_0 V\rho V^* E_0+E_1 V\rho V^* E_1\right]=\mathrm{Pr}_{\rho}^{\mathbf{T}}(0)\cdot\mathbf{T}_0[\rho]+\mathrm{Pr}_{\rho}^{\mathbf{T}}(1)\cdot\mathbf{T}_1[\rho] \tag{4.7}$$

The proof is trivial.

Remark 4.3. $\mathrm{Tr}_{\mathrm{ins}}\left[E_0 V\rho V^* E_0 + E_1 V\rho V^* E_1 \right]$ can be interpreted as a predictable state after testing. We shall denote it by $\mathbf{T}_{\mathrm{total}}[\rho]$. Using statement 4.3 and statement 4.4 we get

$$\mathbf{T}_{\mathrm{pr}}[\rho] = (\mathbf{1}-M)^{1/2}\rho(\mathbf{1}-M)^{1/2} + M^{1/2}\rho M^{1/2} \tag{4.8}$$

Using (4.7) we can rewrite (4.8)

$$\mathbf{T}_{\mathrm{pr}}[\rho] = \mathrm{Pr}_\rho^{\mathbf{T}}(0)\cdot\mathbf{T}_0[\rho] + \mathrm{Pr}_\rho^{\mathbf{T}}(1)\cdot\mathbf{T}_1[\rho] \tag{4.9}$$

5. Conservation of qubit state purity in binary testing

In this section we shall show that a pure qubit state is transforming to a pure qubit state by binary testing.

Statement 5.1. A qubit state density matrix ρ is a pure state density matrix iff

$$\rho = q|0\rangle\langle 0| + e^{i\psi}\sqrt{pq}|0\rangle\langle 1| + e^{-i\psi}\sqrt{pq}|1\rangle\langle 0| + p|1\rangle\langle 1| \tag{5.1}$$

for some $p,q \ge 0$ such that $p+q=1$ is held.

Proof. Let $\xi \in \mathcal{H}_2$ be a unit vector then $\xi = r_0 e^{i\psi_0}|0\rangle + r_1 e^{i\psi_1}|1\rangle$ where $r_0, r_1 \ge 0$ and $r_0^2 + r_1^2 = 1$.

Consider the density matrix $\rho_\xi = |\xi\rangle\langle\xi|$.

$$\rho_\xi = |\xi\rangle\langle\xi| = \left(r_0 e^{i\psi_0}|0\rangle + r_1 e^{i\psi_1}|1\rangle\right)\left(r_0 e^{-i\psi_0}\langle 0| + r_1 e^{-i\psi_1}\langle 1|\right) =$$

$$= r_0^2|0\rangle\langle 0| + e^{i(\psi_0-\psi_1)}r_0 r_1|0\rangle\langle 1| + e^{-i(\psi_0-\psi_1)}r_0 r_1|1\rangle\langle 0| + r_1^2|1\rangle\langle 1|$$

Denote $q = r_0^2$, $p = r_1^2$, $\psi = \psi_0 - \psi_1$ then

$$\rho_\xi = q|0\rangle\langle 0| + e^{i\psi}\sqrt{pq}|0\rangle\langle 1| + e^{-i\psi}\sqrt{pq}|1\rangle\langle 0| + p|1\rangle\langle 1|$$

where $p,q \ge 0$ and $p+q=1$.

Conversely, let ρ be density matrix (5.1). Denote by ξ a unit vector $e^{i\psi}\sqrt{q}|0\rangle + \sqrt{p}|1\rangle$ then $\rho = \rho_\xi$. QED.

Statement 5.2. Let $\mathbf{T} = \{\mathbf{1}-M, M\}$ be a qubit binary test, ρ be a pure qubit state density matrix then $\mathbf{T}_0[\rho]$ and $\mathbf{T}_1[\rho]$ are pure qubit state density matrices.

Proof. Suppose $\{|0\rangle, |1\rangle\}$ is the eigenbasis corresponding with M; and $0 \le m_0 \le m_1 \le 1$ are corresponding eigenvalues; and

$$\rho = q|0\rangle\langle 0| + e^{i\psi}\sqrt{pq}|0\rangle\langle 1| + e^{-i\psi}\sqrt{pq}|1\rangle\langle 0| + p|1\rangle\langle 1|$$

then using (4.6) we get

$$\mathbf{T}_1[\rho] = \frac{qm_0}{qm_0+pm_1}|0\rangle\langle 0| + \frac{e^{i\psi}\sqrt{pqm_0m_1}}{qm_0+pm_1}|0\rangle\langle 1| + \tag{5.2}$$

$$+\frac{e^{-i\psi}\sqrt{pqm_0m_1}}{qm_0+pm_1}|1\rangle\langle 0| + \frac{pm_1}{qm_0+pm_1}|1\rangle\langle 1|$$

Denote by q' the value $\dfrac{qm_0}{qm_0 + pm_1}$ and by p' the value $\dfrac{pm_1}{qm_0 + pm_1}$ then we obtain from (5.2)

$$\mathbf{T}_1[\rho] = q'|0\rangle\langle 0| + e^{i\psi}\sqrt{p'q'}|0\rangle\langle 1| + e^{-i\psi}\sqrt{p'q'}|1\rangle\langle 0| + p'|1\rangle\langle 1|$$

It is evident that $p', q' \geq 0$ and

$$p' + q' = \frac{pm_1}{qm_0 + pm_1} + \frac{qm_0}{qm_0 + pm_1} = 1$$

By Statement 5.1 $\mathbf{T}_1[\rho]$ is a pure qubit density matrix. Similarly, one can prove that $\mathbf{T}_0[\rho]$ is a pure qubit density matrix too. QED.

6. Consecutive qubit binary testing

In this section we shall consider probability distributions on the space of outcome sequences corresponding to consecutive qubit binary testing.

Remind that $\Sigma = \{0,1\}$.

Let Σ^ω be a space of all outcome sequences with Tychonoff topology and generated by it Borelean structure.

Let $\rho = q|0\rangle\langle 0| + e^{i\psi}\sqrt{pq}|0\rangle\langle 1| + e^{-i\psi}\sqrt{pq}|1\rangle\langle 0| + p|1\rangle\langle 1|$ be a pure qubit state density matrix, $\mathbf{T} = \{\mathbf{1} - M, M\}$ be a qubit binary test. We shall describe a probability measure on Σ^ω generated by an infinite sequence of qubit measuring by test $\mathbf{T}$.

Consider an alphabet $\overline{\Sigma} = \{0,1,*\}$. Let $w = s_1 \ldots s_n$ be a word under $\overline{\Sigma}$ and let the following condition be satisfied: $s_n \neq '*'$, then we shall say that the set

$$Z_w = \left\{\sigma \in \Sigma^\omega \mid (\forall i \mid 1 \leq i \leq n \wedge s_i \neq '*')(\sigma_i = s_i)\right\}$$

is a cylindrical set.

As known [12, 13], the family of cylindrical sets generates Borelean structure on Σ^ω and each Borelean measure is uniquely defined by its values on such sets.

<u>Statement 6.1.</u> Let ρ be a pure qubit state density matrix, $\mathbf{T} = \{M_0, M_1\}$ be a qubit binary test, $w = s_1 \ldots s_n$ be a word under alphabet Σ then

$$\Pr\nolimits_\rho^{\mathbf{T}}(Z_w) = \mathrm{Tr}\left(\rho M_{s_1}^{1/2} \ldots M_{s_{n-1}}^{1/2} M_{s_n} M_{s_{n-1}}^{1/2} \ldots M_{s_1}^{1/2}\right) \tag{6.1}$$

Proof. By $\mathbf{T}_{s_1 \ldots s_k}[\rho]$ denote a qubit density matrix after k time measuring by binary test of a qubit with initial state described by a density matrix ρ if outcomes sequence is $s_1 \ldots s_n$.

We claim that

$$\mathbf{T}_{s_1\ldots s_k}[\rho]=\frac{M_{s_k}^{1/2}\ldots M_{s_1}^{1/2}\rho M_{s_1}^{1/2}\ldots M_{s_k}^{1/2}}{\mathrm{Tr}\left(\rho M_{s_1}^{1/2}\ldots M_{s_{k-1}}^{1/2}M_{s_k}M_{s_{k-1}}^{1/2}\ldots M_{s_1}^{1/2}\right)}$$
$$\Pr_\rho^{\mathbf{T}}(s_1\ldots s_k)=\mathrm{Tr}\left(\rho M_{s_1}^{1/2}\ldots M_{s_{k-1}}^{1/2}M_{s_k}M_{s_{k-1}}^{1/2}\ldots M_{s_1}^{1/2}\right) \tag{6.2}$$

by induction.

In the case $k=1$ using (2.1), (4.5) and (4.6) we get

$$\Pr_\rho^{\mathbf{T}}(0)=\mathrm{Tr}(\rho(\mathbf{1}-M)) \qquad \Pr_\rho^{\mathbf{T}}(1)=\mathrm{Tr}(\rho M)$$
$$\mathbf{T}_0[\rho]=\frac{(\mathbf{1}-M)^{1/2}\rho(\mathbf{1}-M)^{1/2}}{\mathrm{Tr}(\rho(\mathbf{1}-M))} \qquad \mathbf{T}_1[\rho]=\frac{M^{1/2}\rho M^{1/2}}{\mathrm{Tr}(\rho M)}$$

Further, if statement is true for all $n\le k$ we obtain

$$\Pr_\rho^{\mathbf{T}}(s_1\ldots s_k s_{k+1})=\Pr_\rho^{\mathbf{T}}(s_{k+1}\mid s_1\ldots s_k)\Pr_\rho^{\mathbf{T}}(s_1\ldots s_k)=$$
$$=\mathrm{Tr}\left(\mathbf{T}_{s_1\ldots s_k}[\rho]M_{s_{k+1}}\right)\mathrm{Tr}\left(\rho M_{s_1}^{1/2}\ldots M_{s_{k-1}}^{1/2}M_{s_k}M_{s_{k-1}}^{1/2}\ldots M_{s_1}^{1/2}\right)=$$
$$=\mathrm{Tr}\left(M_{s_k}^{1/2}\ldots M_{s_1}^{1/2}\rho M_{s_1}^{1/2}\ldots M_{s_k}^{1/2}M_{s_{k+1}}\right)=\mathrm{Tr}\left(\rho M_{s_1}^{1/2}\ldots M_{s_k}^{1/2}M_{s_{k+1}}M_{s_k}^{1/2}\ldots M_{s_1}^{1/2}\right)$$

and

$$\mathbf{T}_{s_1\ldots s_k s_{k+1}}[\rho]=\mathbf{T}_{s_{k+1}}\left[\mathbf{T}_{s_1\ldots s_k}[\rho]\right]=\frac{M_{s_{k+1}}^{1/2}\mathbf{T}_{s_1\ldots s_k}[\rho]M_{s_{k+1}}^{1/2}}{\mathrm{Tr}\left(\mathbf{T}_{s_1\ldots s_k}[\rho]M_{s_{k+1}}\right)}=$$
$$=\frac{\mathrm{Tr}\left(\rho M_{s_1}^{1/2}\ldots M_{s_{k-1}}^{1/2}M_{s_k}M_{s_{k-1}}^{1/2}\ldots M_{s_1}^{1/2}\right)}{\mathrm{Tr}\left(\rho M_{s_1}^{1/2}\ldots M_{s_{k-1}}^{1/2}M_{s_k}M_{s_{k-1}}^{1/2}\ldots M_{s_1}^{1/2}\right)}\cdot\frac{M_{s_{k+1}}^{1/2}M_{s_k}^{1/2}\ldots M_{s_1}^{1/2}\rho M_{s_1}^{1/2}\ldots M_{s_{k+1}}^{1/2}}{\mathrm{Tr}\left(\rho M_{s_1}^{1/2}\ldots M_{s_k}^{1/2}M_{s_{k+1}}M_{s_k}^{1/2}\ldots M_{s_1}^{1/2}\right)}=$$
$$=\frac{M_{s_{k+1}}^{1/2}M_{s_k}^{1/2}\ldots M_{s_1}^{1/2}\rho M_{s_1}^{1/2}\ldots M_{s_{k+1}}^{1/2}}{\mathrm{Tr}\left(\rho M_{s_1}^{1/2}\ldots M_{s_k}^{1/2}M_{s_{k+1}}M_{s_k}^{1/2}\ldots M_{s_1}^{1/2}\right)}$$

This completes the proof of Statement 6.1. QED.

Using Statement 6.1 we compute a probability distribution corresponding with qubit binary testing.

In this case all operators in the product $M_{s_1}^{1/2}\ldots M_{s_k}^{1/2}M_{s_{k+1}}M_{s_k}^{1/2}\ldots M_{s_1}^{1/2}$ commute, therefore $M_{s_1}^{1/2}\ldots M_{s_k}^{1/2}M_{s_{k+1}}M_{s_k}^{1/2}\ldots M_{s_1}^{1/2}=M_0^{k+1-N}M_1^N$ where N is a number of '1' in the sequence $s_1\ldots s_k s_{k+1}$.

Hence,

$$\Pr_\rho^{\mathbf{T}}(Z_w)=\mathrm{Tr}\left(\rho(\mathbf{1}-M)^{|w|-N(w)}M^{N(w)}\right)$$

where $N(w)$ is a number of '1' in the outcomes sequence w and $|w|$ is total number of symbols in w.

Thus we have

$$\Pr_{\rho}^{\mathbf{T}}(Z_w) = q\left(1-m_0\right)^{|w|-N(w)} m_0^{N(w)} + p\left(1-m_1\right)^{|w|-N(w)} m_1^{N(w)} \qquad (6.3)$$

It is evident, that the probability distribution corresponding to a consecutive binary testing of a qubit in a pure state is a binomial distribution mixture.

Summary

The mathematical model of a qubit binary test is described and studied in the paper. The model grounds on the idea that each qubit binary testing is a sequence of three steps: the first, combining qubit with a classical instrument; the second, measuring the combined system by projective measurement with two outcomes; and the third, removing the instrument from the combined system.

Using the model made possible to obtain

1) representation of a qubit state after measurement if an initial state and the test outcome is known (formulas (4.5) and (4.6));

2) representation of a predictable qubit state after measurement if an initial state is known (formula (4.9));

3) proving of the statement about conservation of qubit state purity in binary testing (Statement 5.2);

4) formulas for a probability of outcomes sequence and formula for a posterior qubit state density matrix in a consecutive binary testing if corresponding outcomes sequence is known (formulas (6.2));

5) formula of a probability distribution for consecutive binary testing of a qubit in a pure state (formula (6.3)).

Our further research will deal with studying of stochastic properties of qubit binary testing and generalizing results of the paper for the case of consecutive qubit measuring with three outcomes.

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REFERENCES

1. Feynman R.P. Simulating physics with computers. – Internat. J. Theor. Phys. – № 21, 1982. – P. 467 – 488.
2. Feynman R.P. Quantum mechanical computers. – Found. Phys. – № 16, 1986. – P. 507 – 531.
3. Shor P.W. Algorithms for quantum computation: Discrete logarithms and factoring // Proc. 35th Ann. Symp. on Found. of Comp. Sc. – Los Alamitos, CA: IEEE Computer Society Press, 1994. – P. 124–134.
4. Shor P.W. Polynomial-Time Algorithms for Prime Factorization and Discrete Logarithms on a Quantum Computer // arXiv: quant-ph/9508027 v2, 25.01.96. – 28 p.
5. Grover L.K. Quantum mechanics helps in searching for a needle in a haystack. – Phys. Rev. Lett. – Vol. 79, № 2, 1997. – P. 325 – 331.

6. Harel D., Kozen D., Tiuryn J. Dynamic Logic. – Cambridge, MA: MIT Press, 2000. – 459 p.
7. Davies E.B. Quantum Theory of Open Systems. – London: Academic Press, 1976. – 171 p.
8. Coherent Rabi response of a charge-phase qubit under microwave irradiation / Shnyrkov V.I., Born D., Soroka A.A., Krech W. – Phys. Rev. – V. 79, № 1, 2009. – P. 1 – 12.
9. Nielsen M.A., Chuang I.L. Quantum Computation and Quantum Information. – Cambridge: Cambridge University Press, 2000. – 676 p.
10. Holevo A.S. Statistical Structure of Quantum Theory. – Berlin: Springer-Verlag, 2001. – 315 p.
11. Ахиезер Н.И., Глазман И.М. Теория линейных операторов в гильбертовом пространстве, в 2-х томах. – Х.: Вища школа, 1977.
12. Garding L., Whightman A. Representations of the anticommutation relations. – Proc. Nat. Acad. Sci. USA. – № 40, 1954. – P. 617 – 621.
13. Голодец В.Я. Описание представлений антикоммутационных соотношений. – УМН. – Т. 24, № 4, 1969. – С. 3 – 64.

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